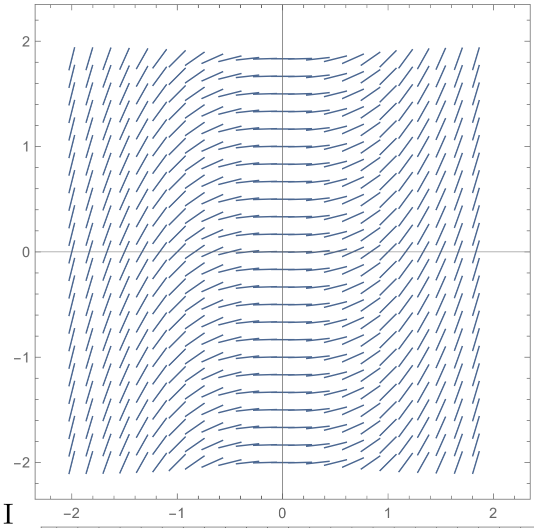
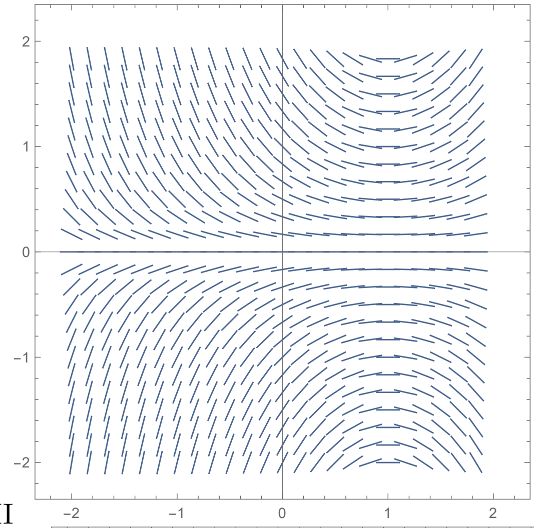
MATH 2300-015 QUIZ 11 (in class) Name: ____________________________

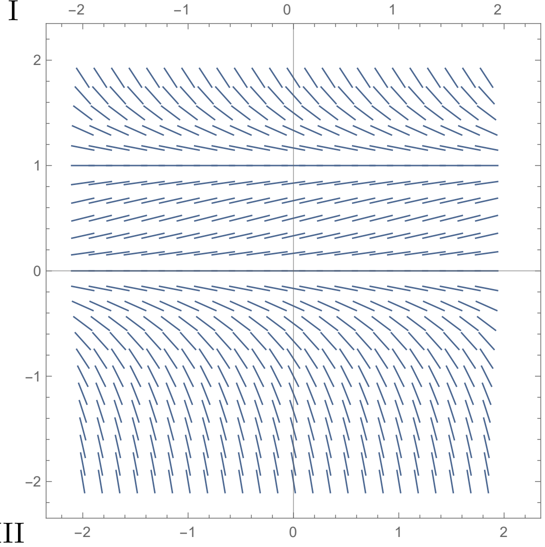
1. Match the slope fields below (labeled I, II, III, IV) with the differential equations below.



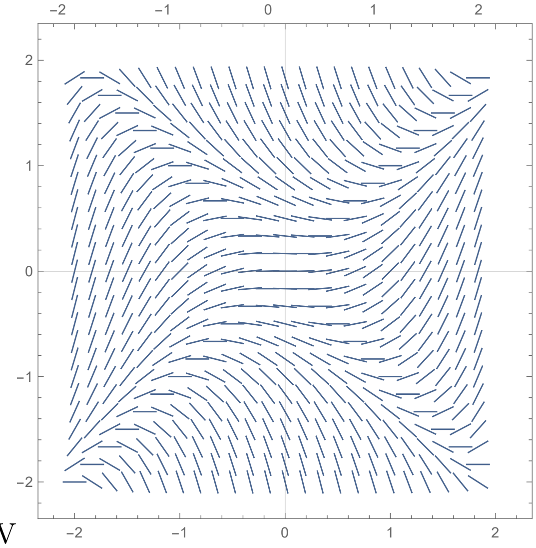
I



II



III



IV

$$\underline{I} \quad \frac{dy}{dx} = x^2$$

$$\underline{III} \quad \frac{dy}{dx} = y - y^2$$

$$\underline{IV} \quad \frac{dy}{dx} = x^2 - y^2$$

$$\underline{II} \quad \frac{dy}{dx} = xy - y$$

2. Use Euler's method with step size 1/2 to approximate $y(2)$ where y is a solution of the initial value problem

$$y' = x - y, \; y(0) = 1,$$

filling in the information in the table below.

n	x_n	y_n	$y'(x_n)$
0	0	1	0-1 =-1
1	1/2	1-1/2 =1/2	1/2-1/2 =0
2	1	1/2+0 =1/2	1-1/2 =1/2
3	3/2	1/2+1/4 =3/4	3/2-3/4 =3/4
4	2	3/4+3/8 =9/8	

Hence $y(2) \approx 9/8$.

3. Using Taylor's inequality, show that the nth degree Taylor polynomial for $\cos x$ converges to $\cos x$ as $n \to \infty$, i.e. show that $\cos x$ is equal to its Taylor series for all x.

Let $f(x) = \cos x$. Then $f^{(n+1)}(x) = \pm \cos x, \pm \sin x$ so that $|f^{(n+1)}(x)| \leq 1$ for any x. Taylor's inequality states that

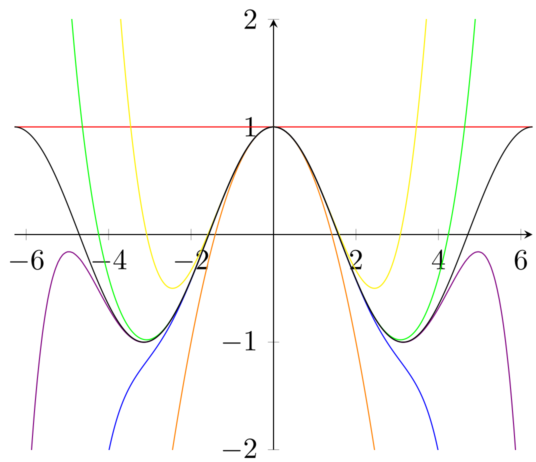
$$|R_n(x)| = |f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x|^{n+1}$$

where M is an upper bound for $|f^{(n+1)}|$ on the interval between 0 and x. We can take $M = 1$ for every n so that for any fixed x we have

$$0 \leq \lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0,$$

showing that the Taylor series $\lim_{n\to\infty} T_n(x)$ converges to $f(x)$.

Below are T_n, $0 \leq n \leq 11$ (red, orange, yellow, green, blue, violet, note that $T_{2n} = T_{2n+1}$) and $\cos x$ (black).



MATH 2300-015 QUIZ 11 Due Tuesday, November 14th Name: ______________________

1. In this problem, you will show that Euler's method converges to an actual solution of the initial value problem below as you take smaller and smaller step sizes.

   (a) Use Euler's method to obtain an estimate $E_n(x)$ of the solution to

   $$y' = y, \; y(0) = 1,$$

   at x by breaking up the interval between 0 and x into n equal pieces.
   We have step size x/n so that after n steps, we reach x. The first few iterations are

   $$\begin{array}{rcl rcl} x_0 &=& 0, & y_0 &=& 1 \\ x_1 &=& x/n, & y_1 &=& 1 + x/n \\ x_2 &=& 2x/n, & y_2 &=& (1 + x/n) + (1 + x/n)x/n = (1 + x/n)^2 \\ x_3 &=& 3x/n, & y_3 &=& (1 + x/n)^2 + (1 + x/n)^2 x/n = (1 + x/n)^3 \\ & \cdots & & & \cdots & \\ x_n &=& nx/n = x, & y_n &=& (1 + x/n)^n = E_n(x). \end{array}$$

   (b) Find the limit as n approaches infinity in your previous answer, i.e. find

   $$E(x) := \lim_{n\to\infty} E_n(x).$$

   We have

   $$E(x) := \lim_{n\to\infty} E_n(x) = \lim_{n\to\infty} (1 + x/n)^n = e^x,$$

   (taking logarithms and applying l'Hopital's rule for instance).

   (c) Show that the limit $E(x)$ above satisfies the initial value problem.
   $E(x) = e^x$ satisfies $E' = E$ and $E(0) = 1$ so it solves the initial value problem.

2. Solve the following initial value problem using power series

   $$y'' + y = 0, \; y(0) = 0, \; y'(0) = 1,$$

   i.e. assume $y = \displaystyle\sum_{n=0}^{\infty} c_n x^n$ is a solution and solve for the c_n recursively.

   If $y = \displaystyle\sum_{n=0}^{\infty} c_n x^n$, then

   $$y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n,$$

   and we are trying to solve

   $$0 = y'' + y = \sum_{n=2}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=0}^{\infty} c_n x^n = \sum_{n=2}^{\infty} [(n+2)(n+1)c_{n+2} + c_n] x^n,$$

subject to $y(0) = 0$, $y'(0) = 1$. For the above to hold, all of the coefficients $(n+2)(n+1)c_{n+2} + c_n$ must be zero, and the initial conditions give us $c_0 = 0$, $c_1 = 1$. Hence

$$
\begin{aligned}
&0 = (0+2)(0+1)c_{0+2} + c_0,\ c_2 = \frac{-c_0}{2 \cdot 1} = 0,\\
&0 = (1+2)(1+1)c_{1+2} + c_1,\ c_3 = \frac{-c_1}{3 \cdot 2} = \frac{-1}{3!},\\
&0 = (2+2)(2+1)c_{2+2} + c_2,\ c_4 = \frac{-c_2}{4 \cdot 3} = 0,\\
&0 = (3+2)(3+1)c_{3+2} + c_3,\ c_5 = \frac{-c_3}{5 \cdot 4} = \frac{1}{5!},\\
&0 = (4+2)(4+1)c_{4+2} + c_4,\ c_6 = \frac{-c_4}{6 \cdot 5} = 0,\\
&\quad \ldots,\\
&0 = (n+2)(n+1)c_{n+2} + c_n,\ c_{n+2} = \frac{-c_n}{(n+2)(n+1)} = 0 \text{ or } \frac{(-1)^n}{n!}.
\end{aligned}
$$

The resulting series is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ which we recognize as $\sin x$.

More generally, the relation $c_{n+2} = \dfrac{-c_n}{(n+2)(n+1)}$ give us

$$
c_{2k} = \frac{(-1)^k c_0}{(2k)!},\ c_{2k+1} = \frac{(-1)^k c_1}{(2k+1)!},
$$

so that the general solution to $y'' + y = 0$ is

$$
y(x) = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + c_1 \frac{(-1)^k x^{2k+1}}{(2k+1)!} = c_0 \cos x + c_1 \sin x.
$$