MATH 2300-015 QUIZ 10 (in class) Name: ____________________________

1. What are the possible intervals of convergence for a general power series $\sum_{n=0}^{\infty} c_n (x-a)^n$?

   [There are six possibilities depending on the radius of convergence.]

   If the radius of convergence is $R = 0$, then the interval of convergence is a single point $\{a\}$. If the radius of convergence is $R = \infty$, then the interval of convergence is $(-\infty, \infty)$. If $0 < R < \infty$ then the power series converges on one of the intervals

$$(a-R, a+R),\ (a-R, a+R],\ [a-R, a+R),\ [a-R, a+R].$$

2. Given a function $f(x)$ that is infinitely differentiable at $x = a$, what is its Taylor series centered at a? [I.e., how do the coefficients depend on f?]

   The Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

   If f can be expressed as a power series near $x = a$, then that power series must be the Taylor series.

3. [Memorization] What are the Taylor series for the following functions (centered at zero)?

   (a) $\sin x$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

   (b) $\cos x$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

   (c) e^x

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

   (d) $\dfrac{1}{1-x}$

$$\sum_{n=0}^{\infty} x^n$$

   (e) $\ln(1+x)$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

MATH 2300-015 QUIZ 10 Due Tuesday, November 7th Name: ______________________

1. For this problem, let $f(x) = (1+x)^{1/3}$

   (a) Find $f'(x)$, $f''(x)$, and $f'''(x)$.
   We have

$$f'(x) = \left(\frac{1}{3}\right)(1+x)^{-2/3}$$
$$f''(x) = \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)(1+x)^{-5/3}$$
$$f'''(x) = \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)(1+x)^{-8/3}.$$

   (b) What is the maximum M of $|f'''(x)|$ on the interval $[0,1]$?
   The function $|f'''(x)| = \frac{10}{27(1+x)^{7/3}}$ is decreasing on the interval $[0,1]$ hence attains its maximum at $x = 0$, $|f'''(0)| = \frac{10}{27} = M$.

   (c) What is $T_2(x)$, the second degree Taylor polynomial for f centered at $x = 0$?
   The second degree Taylor polynomial is

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 1 + \frac{x}{3} - \frac{x^2}{9}.$$

   (d) Use $T_2(x)$ to estimate $\sqrt[3]{2}$.
   $\sqrt[3]{2} = f(1)$ so we estimate using $T_2(1) = 1 + 1/3 - 1/9 = 11/9$.

   (e) Bound the absolute value of the remainder $R_2(1) = f(1) - T_2(1) = \sqrt[3]{2} - T_2(1)$ using Taylor's inequality and the bound M on $|f'''(x)|$ you found above.
   Taylor's inequality states that

$$|R_2(1)| \leq \frac{M}{3!}|1-0|^3 = \frac{5}{81}$$

   so that

$$\frac{94}{81} \leq \sqrt[3]{2} \leq \frac{104}{81}.$$

2. (a) Find $\lim_{x\to 0} \frac{1 - x^2 - e^{-x^2}}{x^4}$ (using a power series representation for e^{-x^2}).
   Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$, we have

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - \frac{x^2}{1} + \frac{x^4}{2} - \frac{x^6}{6} + \dots$$

   so that

$$\frac{1 - x^2 - e^{-x^2}}{x^4} = \frac{1}{x^4}\left(-\frac{x^4}{2} + \frac{x^6}{6} - \dots\right) = -\frac{1}{2} + \frac{x^2}{6} - \dots \to -\frac{1}{2}$$

   as $x \to 0$.

(b) Find

$$\int_0^1 \frac{1 - x^2 - e^{-x^2}}{x^4} dx$$

by integrating a power series term-by-term (your answer will be an infinite series). We have

$$\int \frac{1 - x^2 - e^{-x^2}}{x^4} dx = C + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{(2n-3)n!} x^{2n-3},$$

so that

$$\int_0^1 \frac{1 - x^2 - e^{-x^2}}{x^4} dx = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{(2n-3)n!} = -\frac{1}{2} + \frac{1}{18} - \frac{1}{120} + \frac{1}{840} - \cdots.$$

3. Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n n}{\sqrt{n^3+1}} (x-1)^n$

The radius of convergence R is determined by

$$\frac{1}{R} = \lim_{n\to\infty} \frac{(n+1)2^{n+1}/\sqrt{(n+1)^3+1}}{n2^n/\sqrt{n^3+1}} = \lim_{n\to\infty} 2\frac{n+1}{n}\sqrt{\frac{n^3+1}{(n+1)^3+1}} = 2$$

so that $R = 1/2$. So the power series converges on $(1-1/2, 1+1/2)$ by the ratio test, but we must check convergence at the endpoints $x = 1/2, 3/2$. At $x = 1/2$ the power series evaluates to

$$\sum_{n=0}^{\infty} \frac{(-2)^n n}{\sqrt{n^3+1}} (1/2-1)^n = \sum_{n=0}^{\infty} \frac{n}{\sqrt{n^3+1}}$$

which diverges by limit comparison to $\sum_n \frac{1}{\sqrt{n}}$. At $x = 3/2$ the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+1}}$$

which converges by the alternating series test (note that the derivative of $x/\sqrt{x^3+1}$ is $\frac{2-x^3}{2(x^3+1)^{3/2}}$ which is negative for $x > \sqrt[3]{2}$, so the absolute value of the terms of the series are decreasing to 0). Hence the interval of convergence is $(1/2, 3/2]$.