MATH 2300-005 QUIZ 9 Name: ____________________

This quiz is due Tuesday, October 25th at the beginning of class. Use additional paper as necessary to submit CLEAR and COMPLETE solutions.

1. Find the interval of convergence for the following power series.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n$

   The radius of convergence R is determined by

   $$\frac{1}{R} = \lim_{n\to\infty} \frac{1/(2(n+1))!}{1/(2n)!} = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0$$

   so that $R = \infty$ and the power series converges on the whole real line, $(-\infty, \infty)$. [Remark: This power series agrees with $\cos(\sqrt{x})$ for $x \geq 0$ and gives an "analytic continuation" of $\cos(\sqrt{x})$, i.e. a natural way to make sense of $\cos(\sqrt{x})$ for negative numbers.]

   (b) $\displaystyle\sum_{n=0}^{\infty} n^n (x+1)^n$

   The radius of convergence R is determined by

   $$\frac{1}{R} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} = \lim_{n\to\infty} (n+1)(1+1/n)^n = \infty$$

   so that $R = 0$ and the power series converges only at $x = -1$ where it is centered.

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n n}{\sqrt{n^3+1}} (x-1)^n$

   The radius of convergence R is determined by

   $$\frac{1}{R} = \lim_{n\to\infty} \frac{(n+1)2^{n+1}/\sqrt{(n+1)^3+1}}{n2^n/\sqrt{n^3+1}} = \lim_{n\to\infty} 2\frac{n+1}{n}\sqrt{\frac{n^3+1}{(n+1)^3+1}} = 2$$

   so that $R = 1/2$. So the power series converges on $(1-1/2, 1+1/2)$ by the ratio test, but we must check convergence at the endpoints $x = 1/2, 3/2$. At $x = 1/2$ the power series evaluates to

   $$\sum_{n=0}^{\infty} \frac{(-2)^n n}{\sqrt{n^3+1}} (1/2-1)^n = \sum_{n=0}^{\infty} \frac{n}{\sqrt{n^3+1}}$$

   which diverges by limit comparison to $\sum_n \frac{1}{\sqrt{n}}$. At $x = 3/2$ the series is

   $$\sum_{n=0}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+1}}$$

   which converges by the alternating series test (note that the derivative of $x/\sqrt{x^3+1}$ is $\frac{2-x^3}{2(x^3+1)^{3/2}}$ which is negative for $x > \sqrt[3]{2}$, so the absolute value of the terms of the series are decreasing to 0). Hence the interval of convergence is $(1/2, 3/2]$.

2. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n^n}{n!} x^n$.

The radius of convergence R is determined by

$$\frac{1}{R} = \lim_{n\to\infty} \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \lim_{n\to\infty} (1+1/n)^n = e$$

so that $R = 1/e$.

3. Find the Nth degree Taylor polynomial for $f(x) = -\ln(1-x)$ centered at $x = 0$.

We have (for $n > 0$)

$$f^{(n)}(x) = \frac{(n-1)!}{(1-x)^n}.$$

Evaluating at $x = 0$ gives

$$f^{(n)}(0) = (n-1)! \text{ for } n > 0, \ f(0) = -\ln(1-0) = 0 \text{ for } n = 0.$$

Hence the Nth Taylor polynomial for f centered at $x = 0$ is

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}(x-0)^n = 0 + \sum_{n=1}^{N} \frac{(n-1)!}{n!}(x-0)^n = \sum_{n=1}^{N} \frac{x^n}{n}.$$

For example

$$T_0(x) = 0, \ T_1(x) = x, \ T_2(x) = x + \frac{x^2}{2}, \ T_3(x) = x + \frac{x^2}{2} + \frac{x^3}{3}, \quad \text{etc.}$$