MATH 2300-005 QUIZ 8 Name: ____________________________

1. Determine whether or not the following series converge. Explain your reasoning.

(a) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

We can use the the ratio test or integral test. For the ratio test, we have

$$\lim_{n\to\infty} \frac{(n+1)/e^{(n+1)^2}}{n/e^{n^2}} = \lim_{n\to\infty} \frac{n+1}{n} \frac{e^{n^2}}{e^{(n+1)^2}} = \lim_{n\to\infty} e^{n^2-(n+1)^2} = \lim_{n\to\infty} e^{-2n-1} = 0 < 1.$$

Hence the series converges by the ratio test. For the integral test, note that $f(x) = xe^{-x^2}$ is decreasing

$$f'(x) = -2x^2e^{-x^2} + e^{-x^2} = d^{-x^2}(1 - 2x^1) \leq 0 \text{ for } x \geq 1/\sqrt{2}$$

and we have

$$\int_1^{\infty} xe^{-x^2} dx = -\frac{1}{2} \int_{-1}^{-\infty} e^u du = \frac{1}{2e}$$

so the series converges as well.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+1}$

The series diverges by the limit comparison test. The series $\displaystyle\sum_n \frac{1}{n}$ diverges (the harmonic series, p-series with $p = 1 \geq 1$, integral test, etc.) and we have

$$\lim_{n\to\infty} \frac{(n^2-1)/(n^3+1)}{1/n} = \lim_{n\to\infty} \frac{n^3-n}{n^3+1} = 1$$

so that our original series diverges as well.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2+\sin n}$

The series diverges by the divergence test. We have

$$\frac{1}{2+\sin n} \geq \frac{1}{3}$$

so that the terms of the series are always greater than $1/3$ in absolute value.

2. (a) Show that $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+1}$ converges conditionally.

The series converges by the alternating series test since the absolute value of the alternating terms are decreasing to zero:

$$\frac{d}{dx}\frac{\sqrt{x}}{x+1} = \frac{\frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2}}{(x+1)^2} \leq 0 \text{ for } x \geq 1.$$

However, the series does not converge absolutely by the direct comparison test

$$\frac{\sqrt{n}}{n+1} \geq \frac{1}{\sqrt{n}}$$

and $\sum_{n} \frac{1}{n^{1/2}}$ diverges (p-series, $p = 1/2 \leq 1$).

(b) If $S = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+1}$ is the series above and $S_N = \sum_{n=1}^{N}(-1)^{n-1}\frac{\sqrt{n}}{n+1}$ is the Nth partial sum, find a value of N which guarantees that the absolute value of the remainder $R_N = S - S_N$ is less or equal 0.25.

The alternating series remainder estimate states that

$$\left|\sum_{n=N+1}^{\infty}(-1)^n b_n\right| = |R_N| \leq b_{N+1}.$$

In the case at hande we have

$$\frac{\sqrt{N+1}}{N+2} \leq \frac{1}{4},\ 16(N+1) \leq N^2 + 4N + 4,\ 0 \geq N^2 - 12N - 12$$

so that $N \geq \frac{12+\sqrt{12^2+48}}{2} = 6 + \sqrt{48}$, and we can take $N = 6 + 7 = 13$.