MATH 2300-005 QUIZ 7 Name: ________________________

Due Tuesday, October 11th at the beginning of class. Please use additional paper as necessary to submit CLEAR and COMPLETE solutions.

1. Determine whether or not the following series converge or diverge, using what we have at our disposal thus far (comparison to a known convergent or divergent series or comparison to a convergent or divergent improper integral). Be sure to fully justify your conclusions.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$. The series diverges by comparison to the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. For instance, we have $\displaystyle\lim_{n\to\infty} \frac{1/n}{1/(2n+1)} = 2$ showing that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ both converge or diverge (the *limit comparison test*). One could also use the *integral test* (comparing the series to $\displaystyle\int_1^{\infty} \frac{dx}{2x+1}$, which diverges), or use the *direct comparison test*

   $$\sum_{n=1}^{\infty} \frac{1}{2n+1} \geq \sum_{n=1}^{\infty} \frac{1}{2n+2} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n},$$

   which shows that the series in question is larger than a divergent series.

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1+\sqrt{n}}{\sqrt{n^4-1}}$. We can compare the series to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges (by the integral test or because it is a $p$-series with $p = 3/2 > 1$). Using the limit comparison test, we have

   $$\lim_{n\to\infty} \frac{1+\sqrt{n}}{\sqrt{n^4-1}} \Big/ \frac{1}{n^{3/2}} = \lim_{n\to\infty} \frac{n^2+n^{3/2}}{\sqrt{n^4-1}} = \lim_{n\to\infty} \frac{1+1/\sqrt{n}}{\sqrt{1-1/n^4}} = 1,$$

   so that our series converges as well.

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$. We can use the integral test, comparing the series to the improper integral

   $$\int_1^{\infty} xe^{-x} = -xe^{-x}\Big|_1^{\infty} + \int_1^{\infty} e^{-x}dx = 2/e,$$

   which converges. Be sure to check that $f(x) = x/e^x$ is eventually decreasing: $f'(x) = \frac{1-x}{e^x}$ which is negative for $x > 1$. If we had already covered the ratio test at the time of the quiz, we could have used it here:

   $$\lim_{n\to\infty} \frac{(n+1)/e^{n+1}}{n/e^n} = \lim_{n\to\infty} \frac{n+1}{n}\frac{e^n}{e^{n+1}} = 1/e < 1.$$

   (d) $\displaystyle\sum_{n=1}^{\infty} \sin(1/n)$. We can use the limit comparison test, comparing to the divergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. You should know that $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$. In other words, $\sin(1/n)$ is approximately $1/n$ for large $n$ in that $\displaystyle\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$. Hence the series diverges by the limit comparison test.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n + e^{-n}}{\sqrt{n^3 - n + 1}}$. First note that $0 \leq \dfrac{\sin^2 n + e^{-n}}{\sqrt{n^3 - n + 1}} \leq \dfrac{1 + e^{-n}}{\sqrt{n^3 - n + 1}}$, so that if $\displaystyle\sum_{n=1}^{\infty} \frac{1 + e^{-n}}{\sqrt{n^3 - n + 1}}$ converges, then so does $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n + e^{-n}}{\sqrt{n^3 - n + 1}}$. We can use the limit comparison test, comparing the terms to those of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges ($p$-series, $p = 3/2 > 1$ or use the integral test). We have

$$\lim_{n\to\infty} \frac{\frac{1+e^{-n}}{\sqrt{n^3-n+1}}}{1/n^{3/2}} = \lim_{n\to\infty} (1 + e^{-n}) \frac{n^{3/2}}{\sqrt{n^3 - n + 1}} = 1,$$

so that our original series converges.

2. Evaluate the series $\displaystyle\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$.

The series is telescoping. The $N$th partial sum is

$$\begin{aligned}
&\sum_{n=2}^{N} \ln\left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^{N} (\ln(n+1) + \ln(n-1) - 2\ln n) \\
&= (\ln 3 + \ln 1 - 2\ln 2) + (\ln 4 + \ln 2 - 2\ln 3) + (\ln 5 + \ln 3 - 2\ln 4) \\
&+ \cdots + (\ln(N+1) + \ln(N-1) - 2\ln N) = \ln(N+1) - \ln N - \ln 2 \\
&= \ln\left(\frac{N+1}{N}\right) - \ln 2,
\end{aligned}$$

which converges to $-\ln 2$ as $N \to \infty$.

3. How many terms $N$ of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ can we use to guarantee the remainder

$$R_N = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} - \sum_{n=2}^{N} \frac{1}{n(\ln n)^2}$$

is less than 0.1?

[First note that the series converges by the integral test, which should have been part of the problem.] The remainder $R_N$ is less or equal the improper integral

$$\int_N^{\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln N}^{\infty} \frac{du}{u^2} = \frac{1}{\ln N}.$$

So we want $R_N \leq \frac{1}{\ln N} \leq 0.1$, i.e. $\ln N \geq 10$, $N \geq e^{10}$, $N \geq 22,027$.