MATH 2300-005 QUIZ 6 Name: ______________________

Due Tuesday, October 4th at the beginning of class. Please use additional paper as necessary to submit CLEAR and COMPLETE solutions.

Determine whether the sequence converges or diverges. If it converges, find its limit.

1. $a_n = \dfrac{3^{n+2}}{5^n}$

$$a_n = 9\left(\frac{3}{5}\right)^n \to 0$$

since $|3/5| < 1$.

2. $a_n = \sqrt{\dfrac{n+1}{9n+1}}$

$$a_n = \sqrt{\frac{1+1/n}{9+1/n}} \to \frac{1}{3}$$

since $1/n \to 0$.

3. $a_n = \dfrac{e^n + e^{-n}}{e^{2n} - 1}$

$$a_n = \frac{1+e^{-2n}}{e^n - e^{-n}} \to 0$$

since $e^{-n}, e^{-2n} \to 0$ and $e^n \to \infty$.

4. $a_n = \dfrac{(-3)^n}{n!}$

$$|a_n| = \frac{3^n}{n!} \le \frac{3}{1}\frac{3}{2}\frac{3}{3}\left(\frac{3}{4}\right)^{n-3} \to 0$$

since $|3/4| < 1$, and $a_n \to 0$ as well.

5. $a_n = \ln(2n^2+1) - \ln(n^2+1)$

$$a_n = \ln\left(\frac{2n^2+1}{n^2+1}\right) = \ln\left(\frac{2+1/n^2}{1+1/n^2}\right) \to \ln 2$$

because $\ln x$ is continuous at $x = 2$ and $\frac{2+1/n^2}{1+1/n^2} \to 2$.

6. $a_n = \dfrac{\sin(2n)}{1+\sqrt{n}}$

$$|a_n| \le \frac{1}{1+\sqrt{n}} \to 0$$

so $a_n \to 0$ as well.

Determine whether the series is convergent or divergent. If it converges find the sum.

1.

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1/3}{1-1/3} + \frac{2/3}{1-2/3} = 1/2 + 2 = 5/2.$$

2. $\sum_{n=1}^{\infty} \frac{1+3^n}{2^n}$ The series diverges because the terms of the series do not approach zero. We have $\frac{1+3^n}{2^n} \to \infty$.

3.

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{2}{n^2-1} &= \sum_{n=2}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n+1}\right] \\ &= (1-1/3) + (1/2-1/4) + (1/3-1/5) + (1/4-1/6) + \ldots.\end{aligned}$$

The series is telescoping and the Nth partial sum is $S_N = 1 + \frac{1}{2} - \frac{1}{N} - \frac{1}{N+1}$, which goes to $3/2$ as $N \to \infty$.

4. $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$ The series diverges even though the terms go to zero. We have a telescoping series

$$\begin{aligned}\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) = \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) &= \sum_{n=1}^{\infty} [\ln(n+1) - \ln(n)] \\ &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + \ldots.\end{aligned}$$

The Nth partial sum is $\ln(N+1)$ which goes to infinity as $N \to \infty$.

5. $\sum_{n=1}^{\infty} \frac{n(n+2)}{(n+3)^2}$ The terms of the series do not approach zero; $\frac{n(n+2)}{(n+3)^2} \to 1$ as $n \to \infty$.

6.

$$\sum_{n=1}^{\infty} [(0.8)^{n-1} - (0.3)^n] = \sum_{n=1}^{\infty} (0.8)^{n-1} - \sum_{n=1}^{\infty} (0.3)^n] = \frac{1}{1-0.8} - \frac{0.3}{1-0.3} = 5 - \frac{3}{7} = \frac{32}{7}.$$

1. [section 8.1, problem 55] Let $a_1 = 1$, $a_{n+1} = 3 - \frac{1}{a_n}$. Show that a_n is increasing and bounded above by 3 (so that $\lim_{n\to\infty} a_n$ exists). Find the limit.

   I will be using the "principle of mathematical induction" in what follows. If you want to show a proposition P holds for all natural numbers $n = 0, 1, 2, 3, \ldots$, one can proceed as follows:

   - prove P is true for $n = 0$ (the "base case"),
   - prove that if P is true for n, then it is true for $n + 1$ (the "inductive step").

   From this it follows that P is true for $n = 0$ by the base case, then for $n = 1$ by the induction step since P is true for $n = 0$, then for $n = 2$ by the inductive step since P is true for $n = 1$, etc., and the proposition holds for all $n \geq 0$.

   First note that $a_n \geq 1$ for all n by induction since $a_1 = 1$ and $a_{n+1} = 3 - 1/a_n \geq 1$ holds if and only if $2a_n \geq 1$ which is true assuming $a_n \geq 1$. Secondly, the sequence is increasing by induction since $a_2 - a_1 = 2 - 1 = 1 > 0$ and

   $$a_{n+1} - a_n = (3 - 1/a_n) - (3 - 1/a_{n-1}) = \frac{a_n - a_{n-1}}{a_n a_{n-1}} > 0$$

   assuming $a_n - a_{n-1} > 0$ (the denominator is positive since all terms are greater than or equal to 1). Finally, $a_n < 3$ for all n since we are subtracting a positive number $1/a_{n-1}$ from 3. The bounded increasing sequence a_n thus has a limit L. Taking limits on both sides of $a_{n+1} = 3 - 1/a_n$ gives

   $$L = 3 - 1/L,\ L^2 - 3L + 1 = 0,\ L = \frac{3 \pm \sqrt{5}}{2},$$

   and $L = \frac{3+\sqrt{5}}{2}$ since $L > 1$.

2. [section 8.2, problem 58] See the text for the problem statement.

   The lengths in the picture form a geometric series:

   $$\sum_{n=1}^{\infty} b \sin^n \theta = \frac{b \sin\theta}{1 - \sin\theta}.$$