MATH 2300-005 QUIZ 3 Name: ____________________________

Due Tuesday, September 13th at the beginning of class. Please use additional paper as necessary to submit CLEAR and COMPLETE solutions.

1. We know that $\int_1^\infty \frac{dx}{x}$ diverges but increasing the exponent of x in the denominator by any amount produces a convergent improper integral. Show that the family of functions $\{x(\ln x)^p : p > 0\}$ is "between" x and the family $\{x^p : p > 1\}$ in the following sense:

$$\lim_{x\to\infty} \frac{x(\ln x)^p}{x^q} = 0, \quad \text{for any } q > 1, p > 0.$$

(If you find this too confusing, you may do only the cases $p = 1, 2, 3$.)

As an example, for $p = 2$ we have

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x^{q-1}} \overset{L'H}{=} \lim_{x\to\infty} \frac{2\ln x}{(q-1)x^{q-1}} \overset{L'H}{=} \lim_{x\to\infty} \frac{2}{(q-1)^2 x^{q-1}} = 0.$$

In general, you can use L'Hopital's rule n times, where n is smallest integer greater than p:

$$\lim_{x\to\infty} \frac{(\ln x)^p}{x^{q-1}} = \cdots = \lim_{x\to\infty} \frac{p(p-1)\ldots(p-n+1)(\ln x)^{p-n}}{(q-1)^n x^{q-1}} = 0,$$

since the exponent $p - n$ of $\ln x$ is less than or equal to zero.

2. For which values of p does $\int_e^\infty \frac{dx}{x(\ln x)^p}$ converge/diverge? Find the value of the improper integral when it is convergent.

If $p \neq 1$ we have

$$\begin{aligned}\int_e^\infty \frac{dx}{x(\ln x)^p} &= \lim_{t\to\infty} \int_e^t \frac{dx}{x(\ln x)^p} = \lim_{t\to\infty} \int_1^{\ln t} \frac{du}{u^p} = \lim_{t\to\infty} \frac{u^{1-p}}{1-p}\Big|_1^{\ln t} \\ &= \lim_{t\to\infty} \frac{(\ln t)^{1-p}}{1-p} - \frac{1}{1-p}\end{aligned}$$

which is $\frac{1}{p-1}$ if $p > 1$ and ∞ if $p > 1$. For $p = 1$ we get $\lim_{t\to\infty} \ln(\ln t) = \infty$ and the integral diverges as well. In summary

$$\int_e^\infty \frac{dx}{x(\ln x)^p} = \begin{cases} \infty & p \leq 1 \\ \frac{1}{p-1} & p > 1 \end{cases}.$$

3. For what values of p does the improper integral $\int_0^1 \frac{dx}{x^p}$ converge?

For $p \neq 1$ we have

$$\begin{aligned}\int_0^1 \frac{dx}{x^p} &= \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \frac{x^{1-p}}{1-p}\Big|_t^1 \\ &= \frac{1}{1-p} - \lim_{t\to 0^+} \frac{t^{1-p}}{1-p}\end{aligned}$$

which is $\frac{1}{1-p}$ if $p<1$ and ∞ if $p>1$. For $p=1$ we get $\lim_{t\to 0^+} -\ln t = \infty$ and the integral diverges as well. In summary

$$\int_0^1 \frac{dx}{x^p} = \left\{ \begin{array}{cc} \infty & p \geq 1 \\ \frac{1}{1-p} & p < 1 \end{array} \right. .$$

4. First, show that $\displaystyle\int_0^\infty \frac{dx}{x^3+1}$ converges by comparison. Second, find the value of the improper integral (you should get $\frac{2\pi}{3\sqrt{3}}$).

   For instance, we can compare $\frac{1}{1+x^3} \leq \frac{1}{x^3}$ on $[1,\infty)$ so that

$$\int_0^\infty \frac{dx}{1+x^3} = \int_0^1 \frac{dx}{1+x^3} + \int_1^\infty \frac{dx}{1+x^3} \leq \int_0^1 \frac{dx}{1+x^3} + \int_1^\infty \frac{dx}{x^3} < \infty.$$

   As for the actual value, we use partial fractions:

$$\frac{1}{1+x^3} = \frac{A}{x+1}\frac{Bx+C}{x^2-x+1},\ 1 = (A+B)x^2+(-A+B+C)x+(A+C),\ A = \frac{1}{3},\ B = -\frac{1}{3},\ C = \frac{2}{3}.$$

   Hence the improper integral is

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{3}\int_0^t \left(\frac{1}{x+1} + \frac{-x+2}{x^2-x+1}\right) dx \\
&= \lim_{t\to\infty} \frac{1}{3}\ln|t+1| + \frac{1}{6}\int_0^t \frac{-2x+1}{x^2-x+1}dx + \frac{1}{2}\int_0^t \frac{dx}{x^2-x+1} \\
&= \lim_{t\to\infty} \frac{1}{3}\ln|t+1| - \frac{1}{6}\ln|t^2-t+1| + \frac{1}{2}\int_0^t \frac{dx}{(x-1/2)^2+3/4} \\
&= \lim_{t\to\infty} \frac{1}{3}\ln|t+1| - \frac{1}{6}\ln|t^2-t+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2t-1}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}}\arctan(-1/\sqrt{3}) \\
&= \frac{\pi}{2\sqrt{3}} + \frac{\pi}{6\sqrt{3}} + \lim_{t\to\infty}\frac{1}{3}\ln\left|\frac{t+1}{\sqrt{t^2-t+1}}\right| = \frac{2\pi}{3\sqrt{3}}.
\end{aligned}$$

   In the last line we are using the limit

$$\lim_{t\to\infty}\frac{t+1}{\sqrt{t^2-t+1}} = \lim_{t\to\infty}\frac{1+1/t}{\sqrt{1-1/t+1/t^2}} = 1.$$

5. Find the value of C for which the following improper integral converges and evaluate the integral for this value of C:

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right) dx.$$

   The integral is (with $x = 2\tan\theta$ integrating the first summand)

$$\begin{aligned}
&\lim_{t\to\infty}\int_0^{\arctan(t/2)} \sec\theta\ d\theta - C\ln|t+2| + C\ln 2 \\
&= \lim_{t\to\infty} \ln|\sec(\arctan(t/2)) + \tan(\arctan(t/2))| - C\ln|t+2| + C\ln 2 \\
&= \lim_{t\to\infty} \ln|\frac{1}{2}\sqrt{t^2+4} + t/2| - C\ln|t+2| + C\ln 2 = \lim_{t\to\infty}\ln\left|\frac{\sqrt{t^2+4}+t}{2(t+2)^C}\right| + C\ln 2.
\end{aligned}$$

   Now we see that $C=1$ is the only possibility, else $\frac{\sqrt{t^2+4}+t}{2(t+2)^C}$ goes to 0 or ∞ as $t\to\infty$ and the logarithm will diverge. For $C=1$, the value of the integral is $\ln 2$.