MATH 2300-005 QUIZ 2 Name: ____________________________

Due Tuesday, September 6th at the beginning of class. Please use additional paper as necessary to submit CLEAR and COMPLETE solutions.

Use the partial fraction decomposition to integrate these rational functions.

1. $\int \frac{x^2 - x + 5}{x^2 + x - 6} dx$. The integrand is

$$\frac{x^2 - x + 5}{x^2 + x - 6} = 1 + \frac{-2x + 11}{x^2 + x - 6} = 1 + \frac{-2x + 11}{(x + 3)(x - 2)},$$

so we want the partial fraction decomposition of $\frac{-2x+11}{(x+3)(x-2)}$,

$$\frac{-2x + 11}{(x + 3)(x - 2)} = \frac{A}{x + 3} + \frac{B}{x - 2} \Rightarrow -2x + 11 = A(x - 2) + B(x + 3) = (A + B)x - 2A + 3B$$
$$\Rightarrow -22 = A + B,\ 11 = -2A + 3B \Rightarrow A = -17/5,\ B = 7/5.$$

Hence the integral is

$$\int dx - \frac{17}{5} \int \frac{dx}{x + 3} + \frac{7}{5} \int \frac{dx}{x - 2} = x - \frac{17}{5} \ln |x + 3| + \frac{7}{5} \ln |x - 2|.$$

2. $\int \frac{y^2 + 3}{y^3 - 3y^2 + 3y - 1} dy$. The integrand is

$$\frac{y^2 + 3}{y^3 - 3y^2 + 3y - 1} = \frac{y^2 + 3}{(y - 1)^3} = \frac{A}{y - 1} + \frac{B}{(y - 1)^2} + \frac{C}{(y - 1)^3},$$

and we obtain

$$y^2 + 3 = A(y - 1)^2 + B(y - 1) + C = Ay^2 + (B - 2A)y + A - B + C$$
$$\Rightarrow A = 1,\ B - 2A = 0,\ A - B + C = 3 \Rightarrow A = 1,\ B = 2,\ C = 4.$$

Hence the integral is

$$\int \frac{dy}{y - 1} + \int \frac{2dy}{(y - 1)^2} + \int \frac{4dy}{(y - 1)^3} = \ln |y - 1| - 2(y - 1)^{-1} - 2(y - 1)^{-2}.$$

3. $\int \frac{-z^3 - 26z^2 - 28z - 120}{z^4 - 16} dz$. The integrand is

$$\frac{-z^3 - 26z^2 - 28z - 120}{z^4 - 16} = \frac{-z^3 - 26z^2 - 28z - 120}{(z^2 + 4)(z + 2)(z - 2)} = \frac{Az + B}{z^2 + 4} + \frac{C}{z + 2} + \frac{D}{z - 2}.$$

We must solve

$$-z^3 - 26z^2 - 28z - 120 = (Az + B)(z + 2)(z - 2) + C(z - 2)(z^2 + 4) + D(z + 2)(z^2 + 4)$$
$$= (A + C + D)z^3 + (B - 2C + 2D)z^2 + (-4A + 4C + 4D)z + (-4B - 8C + 8D).$$

More compactly, we can reduce

$$\begin{pmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & -2 & 2 & -26 \\ -4 & 0 & 4 & 4 & -28 \\ 0 & -4 & -8 & 8 & -120 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -9 \end{pmatrix}$$

to get $A = 3$, $B = 2$, $C = 5$, $D = -9$. Hence the integral is

$$\int \frac{3z+2}{z^2+4}dz + \int \frac{5dz}{z+2} + \int \frac{-9dz}{z-2} = \frac{3}{2}\int \frac{2z}{z^2+4}dz + 2\int \frac{dz}{z^2+4} + 5\int \frac{dz}{z+2} - 9\int \frac{dz}{z-2}$$
$$= \frac{3}{2}\ln(z^2+4) + \arctan(z/2) + 5\ln|z+2| - 9\ln|z-2|.$$

4. $\displaystyle\int \frac{dw}{w^2-2w+10}$. Completing the square, the denominator is $(w-1)^2+9$ and the integral is

$$\frac{1}{3}\arctan\left(\frac{x-1}{3}\right).$$

Use an appropriate inverse trigonometric substitution to integrate the following. In the case of an idefinite integral, please simplify compositions of trigonometric and inverse trigonometric functions.

1. $\displaystyle\int_{\sqrt{2}}^{2} \frac{dt}{t^4\sqrt{t^2-1}}$. With $t = \sec\theta$, $dt = \sec\theta\tan\theta\ d\theta$, the integral becomes

$$\int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta\ d\theta}{\sec^4\theta\sqrt{\sec^2\theta-1}} = \int_{\pi/4}^{\pi/3} \cos^3\ d\theta = \int_{\pi/4}^{\pi/3} (1-\sin^2\theta)\cos\theta\ d\theta$$
$$= \int_{1/\sqrt{2}}^{\sqrt{3}/2} (1-u^2)du = u - \frac{u^3}{3}\Big|_{1/\sqrt{2}}^{\sqrt{3}/2} = \frac{\sqrt{3}-\sqrt{2}}{2} - \frac{3^{3/2}-2^{3/2}}{24} = .06026\ldots.$$

2. $\displaystyle\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}dx$. With $x = 4\sin\theta$, $dx = 4\cos\theta\ d\theta$, the integral becomes

$$64\int_0^{\pi/3} \sin^3\theta\ d\theta = 64\int_0^{\pi/3}(1-\cos^2\theta)\sin\theta\ d\theta = -64\int_1^{1/2}(1-u^2)du = \frac{40}{3}.$$

You can also do this integral with a substitution, $u = 16 - x^2$.

3. $\displaystyle\int \frac{z^3}{\sqrt{z^2+1}}dz$. With $z = \tan\theta$, $dz = \sec^2\theta d\theta$, the integral becomes

$$\int \tan^3\theta\sec\theta\ d\theta = \int(\sec^2\theta-1)\sec\theta\tan\theta\ d\theta = \int(u^2-1)du = \frac{u^3}{3} - u = \frac{1}{3}\sec^3\theta - \sec\theta$$
$$= \frac{1}{3}\sec^3(\arctan z) - \sec(\arctan z) = \frac{1}{3}(1+z^2)^{3/2} - \sqrt{1+z^2}.$$

You can also do this integral with a substitution, $u = z^2 + 1$.

4. $\int y^2\sqrt{1-y^2}dy$. With $y=\sin\theta$, $dy=\cos\theta\ d\theta$, the integral becomes

$$\int \sin^2\theta\cos^2\theta\ d\theta = \int\left(\frac{\sin(2\theta)}{2}\right)^2 d\theta = \frac{1}{4}\int\frac{1-\cos(4\theta)}{2}d\theta = \frac{\theta}{8}-\frac{\sin(4\theta)}{32}$$
$$=\frac{1}{8}\arcsin y - \frac{1}{32}(2\sin(2\theta)\cos(2\theta)) = \frac{1}{8}\arcsin y - \frac{1}{32}(2(2\sin\theta\cos\theta)(\cos^2\theta-\sin^2\theta))$$
$$=\frac{1}{8}\arcsin y - \frac{1}{8}y\sqrt{1-y^2}(1-2y^2).$$