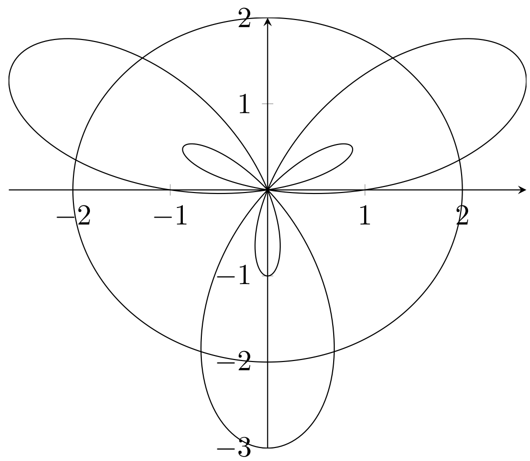
MATH 2300-005 QUIZ 15 Name: ____________________

Due Friday, December 9th at the beginning of class.

1. The polar curves
$$r(\theta) = 1 + 2\sin(3\theta),\ \ r = 2,$$
are graphed below.



(a) Find the area inside the larger loops and outside the smaller loops of the graph of $r = 1 + 2\sin(3\theta)$.

Solution. Solving $r = 0$ gives

$$r = 1+2\sin(3\theta) = 0,\ \sin(3\theta) = -1/2,\ 3\theta = -\pi/6, 7\pi/6+2\pi k,\ \theta = -\pi/18, 7\pi/18+2\pi k/3.$$

One of the "big" loops is traced out by $-\pi/18 \le \theta \le 7\pi/18$, and one of the small loops is traced out by $7\pi/18 \le \theta \le 11\pi/18$. The corresponding areas are

$$\begin{aligned}
A_{big} &= \frac{1}{2}\int_{-\pi/18}^{7\pi/18} (1+2\sin(3\theta))^2 d\theta = \frac{1}{2}\int_{-\pi/18}^{7\pi/18} (1+4\sin(3\theta)+4\sin^2(3\theta))d\theta \\
&= \frac{1}{2}\int_{-\pi/18}^{7\pi/18} (1+4\sin(3\theta)+2(1-\cos(6\theta))d\theta \\
&= \frac{1}{2}\left(3\theta - \frac{4}{3}\cos(3\theta) - \frac{1}{3}\sin(6\theta)\right)\Big|_{-\pi/18}^{7\pi/18} \\
&= \frac{\sqrt{3}}{2} + \frac{2\pi}{3} = 2.96042\ldots,
\end{aligned}$$

$$\begin{aligned}
A_{small} &= \frac{1}{2}\int_{7\pi/18}^{11\pi/18} (1+2\sin(3\theta))^2 d\theta = \frac{1}{2}\int_{7\pi/18}^{11\pi/18} (1+4\sin(3\theta)+4\sin^2(3\theta))d\theta \\
&= \frac{1}{2}\int_{7\pi/18}^{11\pi/18} (1+4\sin(3\theta)+2(1-\cos(6\theta))d\theta \\
&= \frac{1}{2}\left(3\theta - \frac{4}{3}\cos(3\theta) - \frac{1}{3}\sin(6\theta)\right)\Big|_{7\pi/18}^{11\pi/18} \\
&= \frac{\pi}{3} - \frac{\sqrt{3}}{2} = 0.18117\ldots.
\end{aligned}$$

Hence the total area inside the big loops and outside the small loops is

$$3(A_{big} - A_{small}) = \pi + 3\sqrt{3} = 8.3377\ldots.$$

(b) Find the area outside the circle $r = 2$ but inside the curve $r = 1 + 2\sin(3\theta)$.
Solution. We have

$$r = 2 = 1+2\sin(3\theta),\ \sin(3\theta) = 1/2,\ 3\theta = \pi/6, 5\pi/6+2\pi k,\ \theta = \pi/18, 5\pi/18+2\pi k/3.$$

One third of the area outside the circle and inside the other curve is given by

$$\begin{aligned}
\frac{A}{3} &= \frac{1}{2}\int_{\pi/18}^{5\pi/18}[(1+2\sin(3\theta))^2 - 2^2]d\theta = \frac{1}{2}\int_{\pi/18}^{5\pi/18}(-3+4\sin(3\theta)+4\sin^2(3\theta))d\theta \\
&= \int_{\pi/18}^{5\pi/18}(-3/2+2\sin(3\theta)+1-\cos(6\theta))d\theta \\
&= \left(-\frac{\theta}{2} - \frac{2}{3}\cos(3\theta) - \frac{1}{6}\sin(6\theta)\right)\Big|_{\pi/18}^{5\pi/18} \\
&= \frac{5}{2\sqrt{3}} - \frac{\pi}{9} = 1.09430\ldots.
\end{aligned}$$

Hence the total area outside the circle but inside the other curve is

$$A = \frac{5\sqrt{3}}{2} - \frac{\pi}{3} = 3.28292\ldots.$$

(c) What is the tangent line to the curve $r = 1 + 2\sin(3\theta)$ at the point in the first quadrant where r is maximum?
Solution. The maximal value of $r = 1 + 2\sin(3\theta)$ is $r(\pi/6) = 3$, which happens at $(x, y) = (3\cos(\pi/6), 3\sin(\pi/6)) = (3\sqrt{3}/2, 3/2)$. Generally, we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{r\cos\theta + \sin\theta\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}}$$

which with $r = 1 + 2\sin(3\theta)$ gives

$$\frac{dy}{dx} = \frac{(1+2\sin(3\theta))\cos\theta + \sin\theta(6\cos(3\theta))}{-(1+2\sin(3\theta))\sin\theta + \cos\theta(6\cos(3\theta))}.$$

At $\theta = \pi/6$ we have

$$\frac{dy}{dx}\Big|_{\theta=\pi/6} = -\sqrt{3}.$$

Hence the tangent line is

$$y - 3/2 = -\sqrt{3}(x - 3\sqrt{3}/2).$$

(d) Write down a definite integral for the arclength of the curve $r(\theta) = 1 + 2\sin(3\theta)$ and use a computer to evaluate.
Solution. The arclength of a parametric curve $(x(t), y(t))$ for $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

Taking $r = r(\theta)$, $(x(\theta), y(\theta)) = (r\cos\theta, r\sin\theta)$ (the curve is parameterized by θ), we get

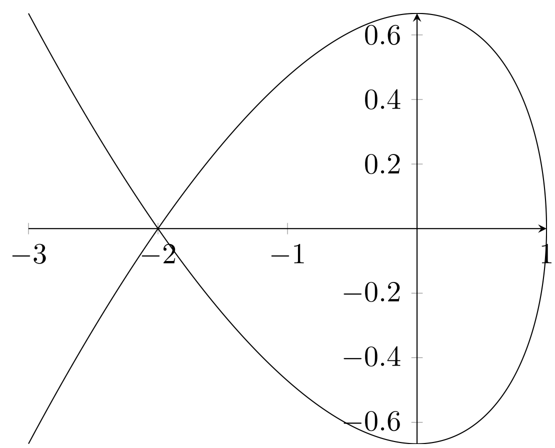
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

In the case at hand, we have

$$L = \int_0^{2\pi} \sqrt{(1 + 2\sin(3\theta))^2 + (6\cos(3\theta))^2}\, d\theta = 27.2667\ldots.$$

2. Consider the parametric curve defined by

$$x(t) = 1 - t^2,\ y(t) = t - t^3/3.$$



(a) Find the equations of the tangent lines to the curve at the point $(-2, 0)$.
Solution. The slope of the tangent line at the point $(x(t), y(t))$ is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - t^2}{-2t}.$$

The t-values corresponding the the point $(-2, 0)$ are $t = \pm\sqrt{3}$ and the corresponding slopes are

$$\left.\frac{dy}{dx}\right|_{t=\pm\sqrt{3}} = \frac{-2}{-2(\pm\sqrt{3})} = \pm\frac{1}{\sqrt{3}}.$$

Hence the tangent lines are

$$y = \frac{x+2}{\sqrt{3}}\ (t = \sqrt{3}),\ y = \frac{-(x+2)}{\sqrt{3}}\ (t = -\sqrt{3}).$$

(b) When/where does the curve have horizontal tangents?
Solution. The slope dy/dx is zero when $dy/dt = 1 - t^2 = 0$, i.e. when $t = \pm 1$. This corresponds to $(x, y) = (0, \pm 2/3)$.

(c) What is the length of the part of the curve forming the "loop"?
Solution. The t-values of the point of self-intersection $(-2,0)$ were found to be $t=\pm\sqrt{3}$ above. The arclength is

$$\begin{aligned} L &= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(-2t)^2+(1-t^2)^2}dt = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4t^2+1-2t^2+t^4}dt \\ &= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(1+t^2)^2}dt = \int_{-\sqrt{3}}^{\sqrt{3}} (1+t^2)dt = t+t^3/3\Big|_{-\sqrt{3}}^{\sqrt{3}} \\ &= 4\sqrt{3} = 6.9282\ldots. \end{aligned}$$

3. [§7.3 exercise 46.] The air in a room of volume 180 m^3 initially contains 0.15% carbon dioxide by volume. Fresh air with a 0.05% concentration (by volume) of carbon dioxide is pumped into the room at a rate of 2 $\mathrm{m}^3/\mathrm{min}$ while well-mixed air is exhausted from the room at the same rate of 2 $\mathrm{m}^3/\mathrm{min}$.

   (a) Write an initial value problem (differential equation and initial condition) describing the rate of change (in $\mathrm{m}^3/\mathrm{min}$) of the volume of carbon dioxide in the room. Then solve the IVP and use this to find the concentration (% by volume) of carbon dioxide in the room as a function of time.
   Solution. Let $A(t)$ denote the volume of CO_2 in the room (measured in m^3) at time t in minutes. Then dA/dt is given by

$$\begin{aligned} \frac{dA}{dt} &= \left(\frac{0.05}{100}\frac{\mathrm{m}^3}{\mathrm{m}^3}\right)\left(2\frac{\mathrm{m}^3}{\mathrm{min}}\right) - \left(\frac{A}{180}\frac{\mathrm{m}^3}{\mathrm{m}^3}\right)\left(2\frac{\mathrm{m}^3}{\mathrm{min}}\right) = \frac{1}{1000} - \frac{A}{90}, \\ A(0) &= \frac{0.15}{100}180 = \frac{27}{100}. \end{aligned}$$

   Separating variables and integrating gives

$$\begin{aligned} \int \frac{dA}{A-9/100} &= -\int \frac{dt}{90} \\ \ln|A-9/100| &= -\frac{t}{90}+C \\ A &= \frac{9}{100}+Ce^{-t/90}, \end{aligned}$$

   and the initial condition $A(0)=0.27$ gives $C=9/50$. If $P(t)$ is the concentration (in % by volume), then $P(t)=(A(t)/180)(100)=5A(t)/9$. Hence

$$P(t) = \frac{1}{20}+\frac{1}{10}e^{-t/90}.$$

   (b) What is the concentration of carbon dioxide in the room after half an hour?
   Solution. $P(30)=1/20+e^{-1/3}/10=0.1216\ldots\%$.

   (c) How long does it take for the concentration of carbon dioxide to reach 0.1%?
   Solution. If $P(t)=0.1=\frac{1}{20}+\frac{1}{10}e^{-t/90}$, then

$$1/2 = e^{-t/90}, \quad t = 90\ln 2 = 62.38\ldots \text{ minutes}.$$

(d) What is the long-term concentration ($t \to \infty$) of carbon dioxide in the room?

Solution. We have

$$\lim_{t \to \infty} P(t) = 1/20 = 0.05\%$$

matching the intuition that we are exhausting the initial air and replacing it with fresh air that has a 0.05% concentration.