MATH 2300-005 QUIZ 13 Name: ____________________________

Due Monday, November 28th at the beginning of class.

1. Recall Newton's law of cooling: the rate of change in temperature of an object is proportional to the difference in temperature between the object and its surroundings,

$$\frac{dT}{dt} = k(T - T_s),$$

where $T(t)$ is temperature as a function of time, k is the proportionality constant, and T_s is the constant surrounding temperature.

Suppose a cup of coffee is 200°F when it is poured and has cooled to 190°F after one minute in a room at 70°F. When will the coffee reach 150°F? What will the temperature of the coffee be after it sits for 30 minutes?

Solution. Separating, integrating, and solving for T gives

$$T(t) = T_s + Ce^{kt}.$$

Using the information provided ($T(0) = 200$, $T(1) = 190$, $T_s = 70$) allows us to solve for k and C

$$200 = T(0) = 70 + C \Rightarrow C = 130,\ \ 190 = T(1) = 70 + 130e^k \Rightarrow k = \ln(12/13).$$

To find when the coffee reaches 150°F, we solve

$$150 = T(t) = 70 + 130e^{kt},\ \ t = \frac{1}{k}\ln(8/13) \approx 6.06 \text{ minutes.}$$

After 30 minutes, the temperature is

$$T(30) = 70 + 130e^{30k} \approx 88.7°\text{F}.$$

2. Differentiate the logistic equation (k, M constant)

$$\frac{dP}{dt} = kP(1 - P/M)$$

with respect to t to get

$$\frac{d^2P}{dt^2} = k^2P(1 - P/M)(1 - 2P/M).$$

What can you deduce from this about solutions $P(t)$?

Solution. Differentiating both sides of the differential equation gives

$$\frac{d}{dt}\left(\frac{dP}{dt}\right) = \frac{d^2P}{dt^2} = kP\left(-\frac{1}{M}\frac{dP}{dt}\right) + (1 - P/M)k\frac{dP}{dt} = k\frac{dP}{dt}\left(1 - \frac{2P}{M}\right)$$

and replacing dP/dt with $kP(1 - P/M)$ on the right gives

$$\frac{d^2P}{dt^2} = k^2P(1 - P/M)(1 - 2P/M).$$

From this we see that every solution of the logistic equation has an inflection point when $P = M/2$, i.e. concavity of solutions changes when the population reaches half of the carrying capacity (marking a change from the early exponential growth to the decay towards the equilibrium solution $P = M$).

3. The following variation on the logistic equation models logistic growth with constant harvesting:

$$\frac{dP}{dt} = kP(1 - P/M) - c.$$

For this problem consider the specific instance

$$\frac{dP}{dt} = 0.08P(1 - P/1000) - 15,$$

modeling fish population in a pond where 15 fish per week are caught (time t in weeks).

(a) What are the equilibrium solutions to the differential equation in part (i.e. what are the constant solutions)?

Solution. The equilibrium solutions are the zeros of the right-hand side of the differential equation, $P = 750, 250$.

(b) Find the general solution of the differential equation. [Integrate using partial fractions. You should get $P(t) = \dfrac{750 - 250Ce^{-t/25}}{1 - Ce^{-t/25}}$ where C is an arbitrary constant.]

Solution. Separating variables, multiplying by a constant, and integrating gives

$$\int \frac{dP}{P^2 - 1000P + 187500} = -\int \frac{dt}{12500}.$$

Partial fractions on the left-hand side gives

$$\frac{1}{P^2 - 1000P + 187500} = \frac{1}{(P - 750)(P - 250)} = \frac{1/500}{P - 750} + \frac{-1/500}{P - 250}.$$

Integrating, we obtain

$$\ln\left|\frac{P - 750}{P - 250}\right| = -\frac{t}{25} + C.$$

Exponentiating gives (different C)

$$\frac{P - 750}{P - 250} = Ce^{-t/25}$$

Finally, solving for P gives the general solution

$$P(t) = \frac{750 - 250Ce^{-t/25}}{1 - Ce^{-t/25}}$$

(c) Find and interpret the solutions with initial conditions $P(0) = 200, 300$.

Solution. We need to solve for C in the following

$$200 = \frac{750 - 250C}{1 - C}, \quad 300 = \frac{750 - 250C}{1 - C}$$

obtaining $C = 11$ and $C = -9$ respectively. So the two particular solutions are

$$P(t) = \frac{750 - 2750e^{-t/25}}{1 - 11e^{-t/25}}, \quad P(t) = \frac{750 + 2250e^{-t/25}}{1 + 9e^{-t/25}}.$$

In the first solution ($P(0) = 200$), the population reaches zero at $t = 25\ln(11/3) \approx 32.48$ weeks, i.e. fishing at a rate of 15 fish/week is unsustainable, while in the second solution ($P(0) = 300$), as $t \to \infty$ the population approaches 750 and the population can sustain this level of fishing.

4. Here's a "heads-up" concerning material for the rest of the semester. You will definitely be responsible for the material in sections 1.7, 3.4, and 6.4 concerning parametric curves, slopes/tangents to parametric curves, and arclength of parametric curves. [You may also want to look at sections 9.1. 9.2, and 9.5 concerning 3D coordinates and vectors, and 10.1, 10.2, 10.3 concerning some simple vector calculus - this is mostly optional.] We will also be covering the material in appendices H.1 and H.2, polar coordinates and arclength/area in polar coordinates.