MATH 2300-005 QUIZ 12 Name: ____________________

1. Find a power series representation (centered at zero) for

$$\frac{1}{(1+x^3)^2} = \frac{d}{dy}\left(\frac{1}{1-y}\right)_{y=-x^3}$$

We have

$$\frac{1}{(1-y)^2} = \frac{d}{dy}\left(\frac{1}{1-y}\right) = \frac{d}{dy}\left(\sum_{n=0}^{\infty} y^n\right) = \sum_{n=1}^{\infty} n y^{n-1},$$

so that

$$\frac{1}{(1+x^3)^2} = \frac{1}{(1-(-x^3))^2} = \sum_{n=1}^{\infty} n(-x^3)^{n-1} = \sum_{k=0}^{\infty}(k+1)(-1)^k x^{3k},$$

(re-indexing $k = n-1$ in the last equality).

2. Solve the following initial value problems.

   (a) $y' + y^2 \sin x = 0,\ y(0) = -1/2$
   Rearranging, we have

$$\frac{dy}{dx} = -y^2 \sin x,\ \frac{dy}{y^2} = -\sin x dx$$

   so that

$$\begin{aligned} \int \frac{dy}{y^2} &= -\int \sin x dx \\ -\frac{1}{y} &= \cos x + C \\ y &= \frac{-1}{\cos x + C}. \end{aligned}$$

   If $y(0) = -1/2$ then $C = 1$ and the solution to the initial value problem is

$$y(x) = \frac{-1}{1+\cos x}.$$

   (b) $y' = \dfrac{x^2}{y(1+x^3)},\ y(0) = -1$
   Separating variables gives

$$y\ dy = \frac{x^2}{1+x^3} dx.$$

   Integrating, we obtain

$$\begin{aligned} \int y\ dy &= \int \frac{x^2}{1+x^3} dx, \\ \frac{y^2}{2} &= \frac{1}{3}\ln|1+x^3| + C, \\ y &= \pm\sqrt{\frac{2}{3}\ln|1+x^3| + C}. \end{aligned}$$

If $y(0) = -1$, then we must have $C = 1$ and the negative square root,

$$y(x) = -\sqrt{\frac{2}{3}\ln|1+x^3|+1}$$

3. Suppose $y(x)$ is the solution to the initial value problem

$$y' = x^2 - y^2, \; y(0) = 1.$$

Use Euler's method (step size 0.1) to approximate $y(0.5)$.

The approximation is $y(0.5) \approx 0.674295419$. The relevant data are in the table below, where $y_{n+1} = y_n + (0.1)(x_n^2 - y_n^2)$:

n	x_n	y_n	$x_n^2 - y_n^2$
0	0	1	-1
1	0.1	0.9	-0.8
2	0.2	0.82	-0.6324
3	0.3	0.75676	-0.482685698
4	0.4	0.70849143	-0.34196106
5	0.5	0.674295419	

4. Use the third degree Taylor polynomial (centered at zero) for $f(x) = \ln(1+x)$ to estimate $\ln(2)$ and use Taylor's inequality to give bounds on the error.

The first four derivatives of $f(x) = \ln(1+x)$ are

$$f'(x) = \frac{1}{1+x}, \; f''(x) = \frac{-1}{(1+x)^2}, \; f'''(x) = \frac{2}{(1+x)^3}, \; f^{(4)}(x) = \frac{-6}{(1+x)^4}.$$

The third degree Taylor polynomial centered at zero is

$$T_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}.$$

To use $T_3(x)$ to approximate $\ln(2)$ we take $x = 1$, $\ln(2) \approx T_3(1) = 1 - 1/2 + 1/3 = 5/6$. A bound for the absolute value of the fourth derivative $f^{(4)}(x)$ on the interval $[0,1]$ is

$$|f^{(4)}(x)| = \frac{-6}{(1+x)^4} |\leq 6 = M$$

and Taylor's inequality states that

$$|\ln(2) - T_3(1)| = |R_n(1)| \leq \frac{M}{(3+1)!}|1-0|^{3+1} = \frac{1}{4}.$$

Hence

$$7/12 = 5/6 - 1/4 \leq \ln(2) \leq 5/6 + 1/4 = 13/12.$$