MATH 2300-005 QUIZ 10 Name: ______________________

This quiz is due Tuesday, November 1st at the beginning of class. Use additional paper as necessary to submit CLEAR and COMPLETE solutions.

1. Find $\lim_{x\to 0} \dfrac{1-x^2-e^{-x^2}}{x^4}$ (using a power series representation for e^{-x^2}). [You may assume e^x is equal to its Taylor series.]

   Since $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ for all $x \in \mathbb{R}$, we have

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - \frac{x^2}{1} + \frac{x^4}{2} - \frac{x^6}{6} + \dots$$

   so that

$$\frac{1-x^2-e^{-x^2}}{x^4} = \frac{1}{x^4}\left(-\frac{x^4}{2} + \frac{x^6}{6} - \dots\right) = -\frac{1}{2} + \frac{x^2}{6} + \dots \to -\frac{1}{2}$$

   as $n \to \infty$.

2. Find

$$\int_0^1 \frac{1-x^2-e^{-x^2}}{x^4} dx$$

   by integrating a power series term-by-term.

   From the previous problem we have

$$\frac{1-x^2-e^{-x^2}}{x^4} = \sum_{n=2}^{\infty} (-1)^{n+1} \frac{x^{2n-4}}{n!}$$

   (the $n = 0, 1$ terms canceled and we divided by x^4). Integrating term-by-term gives

$$\int_0^1 \frac{1-x^2-e^{-x^2}}{x^4} dx = \int_0^1 \left(\sum_{n=2}^{\infty} (-1)^{n+1} \frac{x^{2n-4}}{n!}\right) dx = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n!} \int_0^1 x^{2n-4} dx$$
$$= \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{x^{2n-3}}{2n-3}\Big|_0^1 = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{1}{2n-3}.$$

3. Use the alternating series remainder estimate to to give an approximation to the above integral so that the error is less than 0.001.

   The series $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n!(2n-3)}$ is alternating with the absolute value of the terms decreasing to zero. The absolute value of the nth remainder is bounded by the absolute value of the $(n+1)$st term

$$|R_n| \leq \frac{1}{(n+1)!(2(n+1)-3)}.$$

   The first value of n such that the above is less or equal $1/1000$ is $n = 5$ with

$$\frac{1}{(5+1)!(2(5)-1)} = \frac{1}{6480} < \frac{1}{1000}.$$

   Hence the approximation to the integral is

$$\sum_{n=2}^{5} \frac{(-1)^{n+1}}{n!(2n-3)} = \frac{-569}{1260} = -0.4515873\ldots$$

   whereas the value of the series is $-0.4517253\ldots$, good to the third decimal place.

4. Find the Taylor series centered at zero for the function $f(x) = (1-x^2)^{-1/2}$. [You may assume the results of the text on the binomial series, cf. §8.7, pg. 611–612).

   The Taylor series for $g(x) = (1+x)^{-1/2}$ is given by (and is equal to within the radius of convergence $R = 1$)

$$(1+x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n$$

   where

$$\binom{-1/2}{n} = \frac{(-1/2)(-3/2)(-5/2)\cdot \cdots \cdot(-1/2-n+1)}{n!}, \quad \binom{-1/2}{0} = 1.$$

   Evaluating at $-x^2$ gives

$$(1-x^2)^{-1/2} = g(-x^2) = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^{2n}(-1)^n.$$

5. Integrate the power series from the previous problem term-by-term to find a power series representation for $\arcsin x$ around $x = 0$.

   We know that $\arcsin x = \int \frac{dx}{\sqrt{1-x^2}}$. Integrating the power series from the previous problem term-by-term gives

$$\begin{aligned}\arcsin x &= \int \frac{dx}{\sqrt{1-x^2}} = \int \left(\sum_{n=0}^{\infty} \binom{-1/2}{n} x^{2n}(-1)^n \right) dx = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n \int x^{2n} dx \\ &= \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n \frac{x^{2n+1}}{2n+1} + C\end{aligned}$$

   The constant of integration is seen to be $C = 0$ by evalutating the series and $\arcsin x$ at $x = 0$. This power series representation of $\arcsin x$ is valid within the radius of convergence, i.e. on $(-1, 1)$.

6. (Optional "fun" problem) Suppose

$$f(x) = \frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} c_n x^n.$$

Multiply both sides by $1-x-x^2$ and equate powers of x to show that $c_n = F_n$, the nth Fibonacci number ($F_0 = 1, F_1 = 1, F_n = F_{n-1} + F_{n-2}$). Use partial fractions to write

$$f(x) = \frac{A}{x+\phi} + \frac{B}{x+\bar{\phi}}$$

where $\phi = \frac{1+\sqrt{5}}{2}$, $\bar{\phi} = \frac{1-\sqrt{5}}{2}$. Write $f(x)$ as a power series using the geometric series and the partial fraction decomposition. Finally, give a closed-form expression for the F_n. [You should get something equivalent to $F_n = \frac{(-1)^n}{\sqrt{5}}\left(\frac{1}{\phi^{n+1}} - \frac{1}{\bar{\phi}^{n+1}}\right) = \frac{\phi^{n+1} - \bar{\phi}^{n+1}}{\sqrt{5}}$.]

If

$$\frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} c_n x^n$$

then

$$1 = (1-x-x^2)\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n(x^n - x^{n+1} - x^{n+2}) = c_0 + (c_1 - c_0)x + \sum_{n=2}^{\infty}(c_n - c_{n-1} - c_{n-2})x^n.$$

Equating powers of x on both sides gives

$$c_0 = 1,\ c_0 - c_1 = 0,\ c_n - c_{n-1} - c_{n-2} = 0,\ n \geq 2.$$

Hence $c_0 = c_1 = 1$ and $c_n = c_{n-1} + c_{n-2}$, giving $c_n = F_n$, the nth Fibonacci number.

We have the factorization $1 - x - x^2 = -(x-\phi)(x-\bar{\phi})$ using the quadratic equation. Partial fractions gives

$$\frac{1}{1-x-x^2} = \frac{A}{x+\phi} + \frac{B}{x+\bar{\phi}},\ -1 = (A+B)x + A\bar{\phi} + B\phi,\ A = -B = \frac{1}{\phi - \bar{\phi}} = \frac{1}{\sqrt{5}}.$$

Using the geometric series twice we have

$$\begin{aligned}
\frac{1}{1-x-x^2} &= \frac{1}{\phi\sqrt{5}}\left(\frac{1}{1+x/\phi}\right) - \frac{1}{\bar{\phi}\sqrt{5}}\left(\frac{1}{1+x/\bar{\phi}}\right) \\
&= \frac{1}{\phi\sqrt{5}}\sum_{n=0}^{\infty}(-1)^n(x/\phi)^n - \frac{1}{\bar{\phi}\sqrt{5}}\sum_{n=0}^{\infty}(-1)^n(x/\bar{\phi})^n \\
&= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty} x^n\left(\frac{1}{\phi^{n+1}} - \frac{1}{\bar{\phi}^{n+1}}\right)(-1)^n.
\end{aligned}$$

Equating the two series we've obtained for $\frac{1}{1-x-x^2}$ gives

$$F_n = \frac{(-1)^n}{\sqrt{5}}\left(\frac{1}{\phi^{n+1}} - \frac{1}{\bar{\phi}^{n+1}}\right) = \frac{\phi^{n+1} - \bar{\phi}^{n+1}}{\sqrt{5}}.$$