1 Review

You should be comfortable with everything below (and if you aren't you'd better brush up).

1.1 Arithmetic

You should know how to add, subtract, multiply, divide, and work with the integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ and rational numbers $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$. For instance, we have the **fundamental theorem of arithmetic**: every integer can be written uniquely as a product of primes $(2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, ...)$, e.g.

$$69 = 3 \cdot 23,\ \ 420 = 2^2 \cdot 3 \cdot 5 \cdot 7,\ \ 1065023 = 1031 \cdot 1033.$$

We have the **division algorithm**: given integers n, d, there are q and $r, 0 \leq r < d$ such that $n = qd + r$. This is just grade school division,

$$7/2 = 3 + 1/2, 52/11 = 4 + 8/11, 365/7 = 52 + 1/7, \text{etc.}$$

We have definitions for addition and multiplication of fractions:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd},\ \frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd},\ \frac{a/b}{c/d} = \frac{ad}{bc}.$$

1.2 Functions, Notation, Nomenclature

The basic object of this course is a function $f : I \to \mathbb{R}$, where I is an interval (or finite union of intervals) in $\mathbb{R}$. We will use interval notation, e.g.

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}, [a, b) = \{x \in \mathbb{R} : a \leq x < b\}.$$

The **domain** of a functions is the set where that funciton is defined (the input) and the **range** of a function is the set of values the function takes (the output).

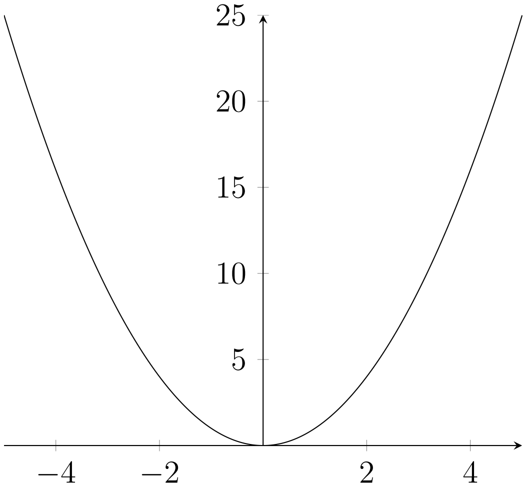
The functions we will be working with in the course are basically resricted to - polynomials, rational funtions, exponential and logarithmic functions, power functions, and compositions thereof.

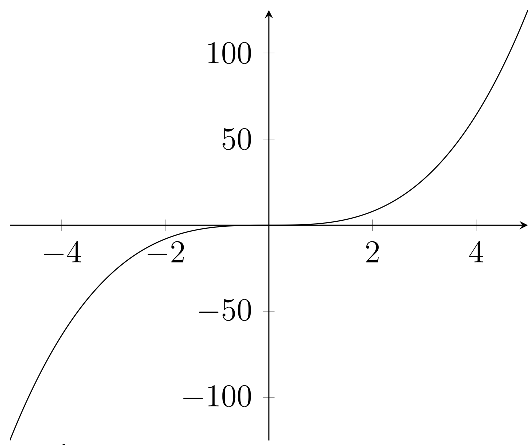
Given a function f such that $f(x) = f(y)$ implies $x = y$ (i.e. the function is one-to-one or invertible), we have the **inverse function** f^{-1} (not to be confused with $1/f$) defined by $f^{-1}(x) = y$ if and only if $f(y) = x$. In other words, $f(f^{-1}(x)) = f^{-1}(f(x)) = x$. For example:

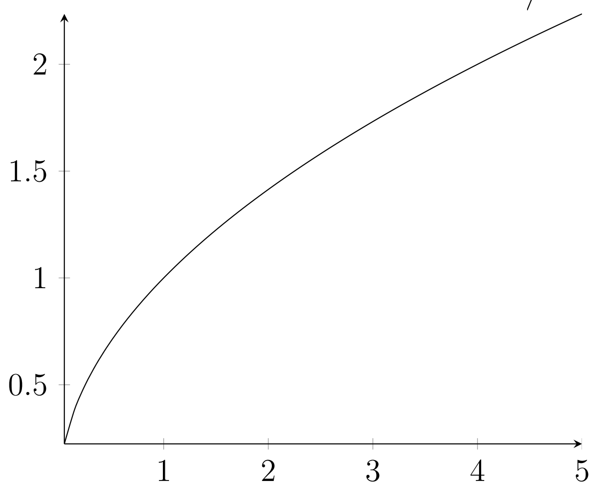
$$f(x) = 3x + 2,\ f^{-1}(x) = \frac{x-2}{3}$$
$$f(x) = (x+2)^2,\ f^{-1}(x) = \sqrt{x} - 2$$
$$f(x) = 2^x,\ f^{-1}(x) = \log_2 x.$$

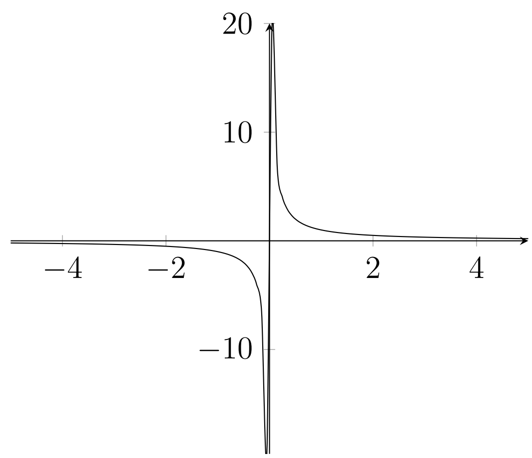
1.3 Power Functions

A power function is a function of the form $f(x) = x^a$ for some $a \in \mathbb{R}$. You should be familiar with their behavior (a odd/even, positive/negative,$0 < a < 1/ \ 1 < a < \infty$, etc.) - for instance: $x^2, x^3, \sqrt{x} = x^{1/2}, x^{-1} = 1/x$





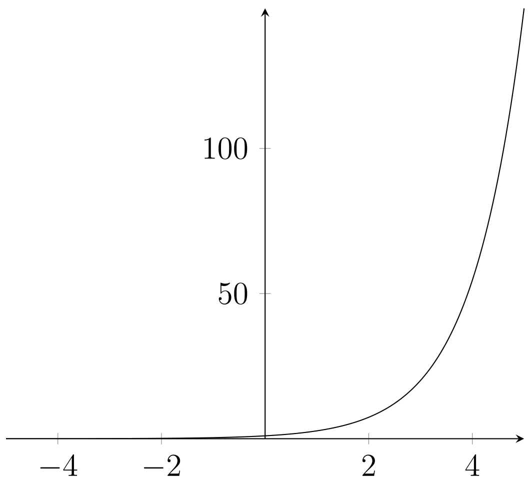


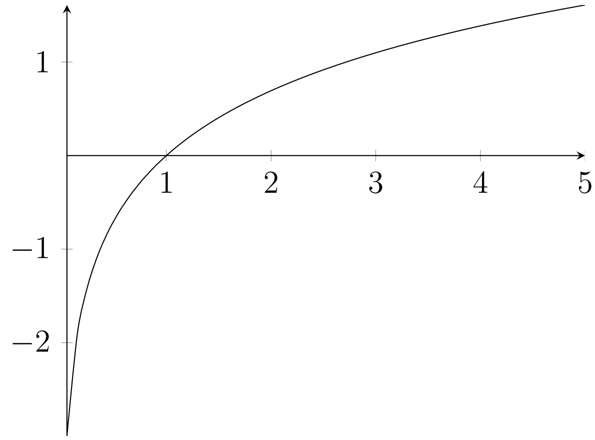


1.4 Exponential and Logarithmic Functions

Here are the basic properties of the exponential and logarithmic functions $b^x : \mathbb{R} \to (0, \infty)$ $\log_b(x) : (0, \infty) \to \mathbb{R}$ (for a fixed base $b > 0$):

$$\begin{aligned} b^{x+y} &= b^x b^y & \log_b(xy) &= \log_b(x) + \log_b(y) \\ b^{x-y} &= b^x / b^y & \log_b(x/y) &= \log_b(x) - \log_b(y) \\ (b^x)^y &= b^{xy} & \log_b(x^y) &= y \log_b(x) \\ b^0 &= 1 & \log_b(1) &= 0 \\ b^1 &= b & \log_b(b) &= 1 \end{aligned}$$





The exponential function b^x and logarithmic function $\log_b(x)$ are inverse to one another

$$b^{\log_b(x)} = x = \log_b(b^x)$$

so to get something out of an exponent, you take a logarithm and to get rid of a logarithm, you exponentiate.

1.5 Lines

A line is determined by any any two distinct points $(x_1, y_1), (x_2, y_2)$ on the line. The ratio $m = (y_2 - y_1)/(x_2 - x_1)$ ("rise over run") is constant and called the **slope** of the line. So the equation of a line through a point (x_0, y_0) with slope m is given by

$$y - y_0 = m(x - x_0).$$

If the line is horizontal, $m = 0$ and we get $y = y_0$ (i.e. y is constant and x takes on every value) and if the line is vertical, the slope is infinite/undefined and we get $x = x_0$ (i.e. x is constant and y takes on every value).

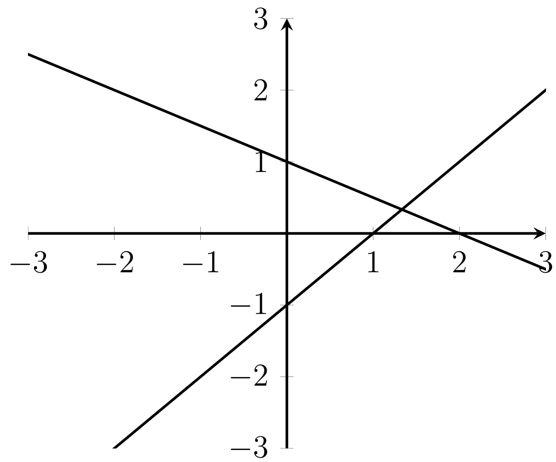
A line with slope m going through the point $(0, b)$ (the y-intercept) has equation (plugging into the above)

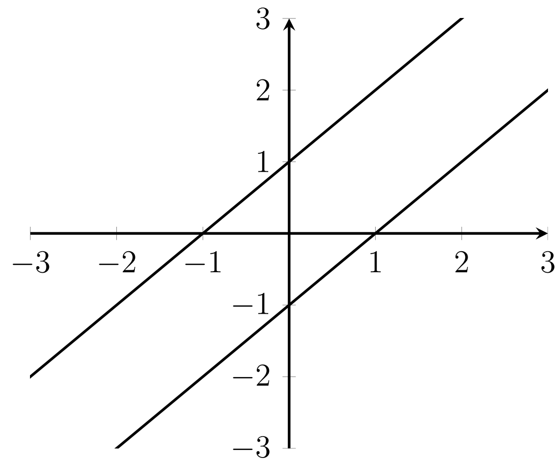
$$y = mx + b.$$

We may need to solve a system of linear equations (two equations in two variables). In general, two lines

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

will have a unique point of intersection $(x, y) = (x_0, y_0)$ - unless the lines are parallel.





1.6 Quadratic polynomials

To find solutions to $ax^2 + bx + c = 0$, factor out a, complete the square, and solve for x:

$$\begin{aligned}
0 &= ax^2 + bx + c \\
&= a\Big(x^2 + \frac{b}{a}x + \frac{c}{a}\Big) \\
&= a\Big(x^2 + \frac{b}{a}x + \Big(\frac{b}{2a}\Big)^2 - \Big(\frac{b}{2a}\Big)^2 + \frac{c}{a}\Big) \\
&= a\Big(\Big(x + \frac{b}{2a}\Big)^2 - \Big(\frac{b}{2a}\Big)^2 + \frac{c}{a}\Big), \\
\Big(\frac{b}{2a}\Big)^2 - \frac{c}{a} &= \Big(x + \frac{b}{2a}\Big)^2, \\
x &= -\frac{b}{2a} \pm \sqrt{\Big(\frac{b}{2a}\Big)^2 - \frac{c}{a}} \\
&= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\end{aligned}$$

This last line is often called the **quadratic formula**.

To factor $x^2 + bx + c = (x - \alpha)(x - \beta)$, note that

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$$

so that

$$b = -(\alpha + \beta),\ c = \alpha\beta.$$

A factorization that comes up frequently is the **difference of squares**

$$x^2 - a^2 = (x - a)(x + a).$$

1.7 Polynomials

A polynomial is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0, a_i \in \mathbb{R}.$$

The integer n is called the **degree** of $p(x)$, the number a_n is the **leading coefficient**.

We have the **fundamental theorem of algebra**: a polynomial always factors (over the complex numbers $\mathbb{C}$) as a product of linear factors, (and over $\mathbb{R}$ as a product of linear and irreducible quadratic factors). We can add, subtract, multiply, and divide polynomials much like integers, for example

$$\begin{aligned}(3x^2+4x+2)+(x^3+x+1) &= x^3+3x^2+(4+1)x+(1+2) = x^3+3x^2+5x+3,\\ (3x^2+4x+2)(x^3+x+1) &= 3x^5+4x^4+(2+3)x^3+(3+4)x^2+(2+4)x+2\\ &= 3x^5+4x^4+5x^3+7x^2+6x+2.\end{aligned}$$

We have a **division algorithm** for polynomials as well, given polynomials n, d there are polynomials q and r, with $r = 0$ or $0 \le \deg(r) < \deg(d)$ such that $n = qd + r$. For example

$$\frac{x^4+3x+2}{x^2+1} = x^2 - 1 + \frac{3x+3}{x^2+1}$$

If $p(c) = 0$, then $(x-c)$ divides $p(x)$, i.e., there is a polynomial $q(x)$ such that $p(x) = (x-c)q(x)$, for example

$$x^2 - 1 = (x-1)(x+1), 1^2 - 1 = 0 = (-1)^2 - 1.$$

We have the **rational roots theorem**: if $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ is a polynomial with integer coefficients, and a/b is a rational root ($p(a/b) = 0$), then $a|a_0, b|a_n$.

We have the **binomial theorem**:

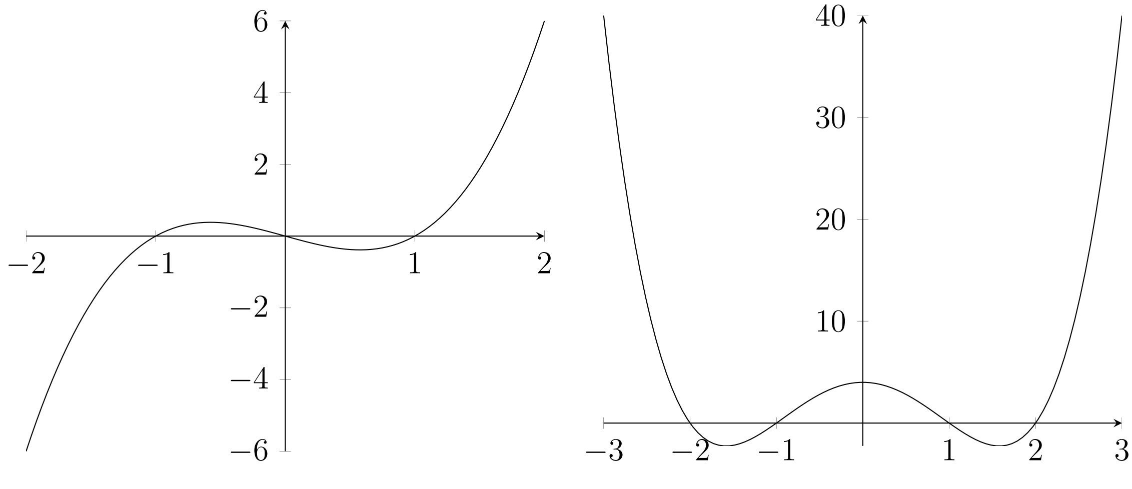
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $k! = 1 \cdot 2 \cdot \cdots \cdot k$. For example:

$$\begin{aligned}(x+y)^0 &= 1\\ (x+y)^1 &= x+y\\ (x+y)^2 &= x^2+2xy+y^2\\ (x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\ (x+y)^4 &= x^4+6x^3y+4x^2y^2+6xy^3+y^4\\ &\text{etc.}\end{aligned}$$

The binomial coefficients can be put into an array known as "Pascal's triangle" via $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Here are the graphs of some polynomials ($x^3 - x, x^4 - 5x^2 + 4$)

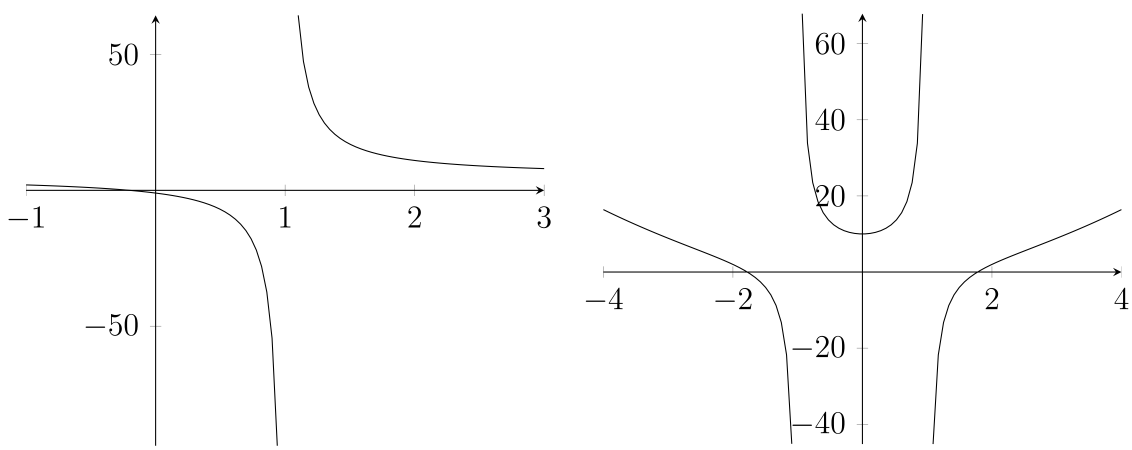


1.8 Rational Functions

A rational function is a quotient of polynomials, $r(x) = p(x)/q(x)$, with domain $\mathbb{R}\backslash\{x : q(x) = 0\}$ (the denominator cannot be zero). We have **vertical asymptotes** of $x = c$ where $q(c) = 0$ and **horizontal asymptotes** of $y = a_n/b_n$ if p and q have the same degree n with leading coefficients a_n and b_n respectively or $y = 0$ if the degree of p is less than the degree of q. If the degree of p is larger than the degree of q, then

$$\lim_{x\to\pm\infty} r(x) = \pm\infty.$$

Examples ($\frac{5x+1}{x-1}, \frac{x^4-3x^2-3}{x^2-1}$)



1.9 Basic Geometry/Analytic Geometry

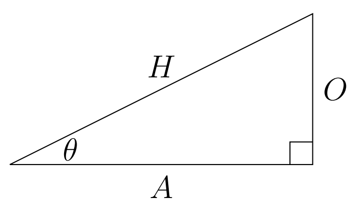
You need to know facts about triangles, circles, cones, cylinders, spheres, etc. For example:

$$\begin{aligned}
\text{area of triangle} &= \frac{1}{2} \times \text{base} \times \text{height} \\
\text{circle area} &= \pi r^2 \\
\text{circle circumfrence} &= 2\pi r \\
\text{volume of sphere} &= \frac{4}{3}\pi r^3 \\
\text{surface area of sphere} &= 4\pi r^2 \\
\text{volume of cylinder} &= \text{area of base} \times \text{height} \\
\text{surface area of cylinder} &= 2 \times \text{area of base} + \text{perimeter of base} \times \text{height} \\
\text{volume of cone} &= \frac{1}{3} \times \text{area of base} \times \text{height} \\
\text{circle of radius } r \ \text{ and center } (h,k) &: (x-h)^2 + (y-k)^2 = r^2 \\
\text{etc.}
\end{aligned}$$

1.10 Trigonometry

Given a right triangle and angle θ in the triangle, we define six functions by ratios of lengths

$$\sin\theta = \frac{O}{H},\ \cos\theta = \frac{A}{H},\ \tan\theta = \frac{O}{A} = \frac{\sin\theta}{\cos\theta},$$
$$\csc\theta = \frac{H}{O} = \frac{1}{\sin\theta},\ \sec\theta = \frac{H}{A} = \frac{1}{\cos\theta},\ \cot\theta = \frac{A}{O} = \frac{1}{\tan\theta}.$$

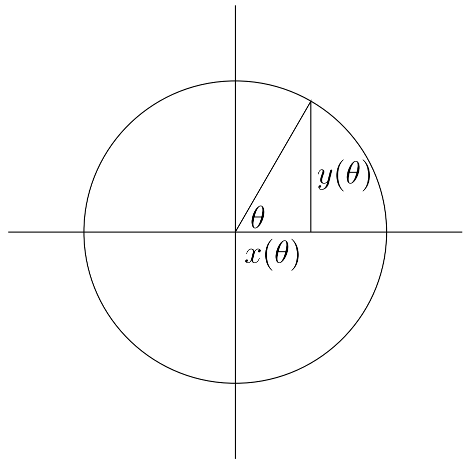


Since these are defined by ratios of lengths, we can fix the hypotheneuse to be 1 and consider all right triangles simultaneously on the unit circle, $x^2 + y^2 = 1$. We then define the trigonometric functions as functions of a real variable (where defined)

$$\sin\theta = y(\theta),\ \cos\theta = x(\theta),\ \tan\theta = \frac{y(\theta)}{x(\theta)} = \frac{\sin\theta}{\cos\theta},$$

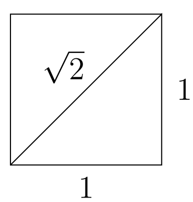
$$\csc\theta = \frac{1}{y(\theta)} = \frac{1}{\sin\theta},\ \sec\theta = \frac{1}{x(\theta)} = \frac{1}{\cos\theta},\ \cot\theta = \frac{x(\theta)}{y(\theta)} = \frac{1}{\tan\theta},$$

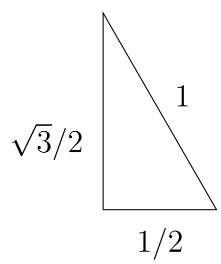
where $(x(\theta), y(\theta))$ are the coordinates of the point on the unit circle at an angle of θ measured counter-clockwise from the positive x-axis.



We measure angles in radians, one radian is the angle subtended by a circular arc of length equal to the radius of the circle ($\approx 57.3°$). There are 2π radians in one circle. By considering an equilateral triangle and a square, we can deduce

$$\cos(\pi/4) = \cos(\pi/4) = \sqrt{2}/2,\ \cos(\pi/3) = \sin(\pi/6) = 1/2,\ \cos(\pi/6) = \sin(\pi/3) = \sqrt{3}/2:$$





We have the following formulas for the sine, cosine, and tangent of a sum or difference of two angles

$$\sin(x \pm y) = \cos x \cos y \pm \sin x \sin y,\ \cos(x \pm y) = \sin x \cos y \mp \cos x \sin y,$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$

You should also know the law of cosines (think of it as the pythagorean theorem for non-right triangles)

$$a^2 + b^2 = c^2 + 2ab\cos\theta$$

where a, b, c are the lengths of the sides and θ is the angle between sides a, b.