Quiz 2, MATH 1300-401

2.3/18

$$\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}=\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}=\lim_{h\to 0}\frac{1+h-1}{h(\sqrt{1+h}+1)}$$
$$=\lim_{h\to 0}\frac{h}{h(\sqrt{1+h}+1)}=\lim_{h\to 0}\frac{1}{\sqrt{1+h}+1}=\frac{1}{\sqrt{1+0}+1}=\frac{1}{2}$$

2.3/19

$$\lim_{x\to -4}\frac{1/4+1/x}{x+4}=\lim_{x\to -4}\frac{x+4}{4x(x+4)}=\lim_{x\to -4}\frac{1}{4x}=\frac{-1}{16}$$

2.3/24

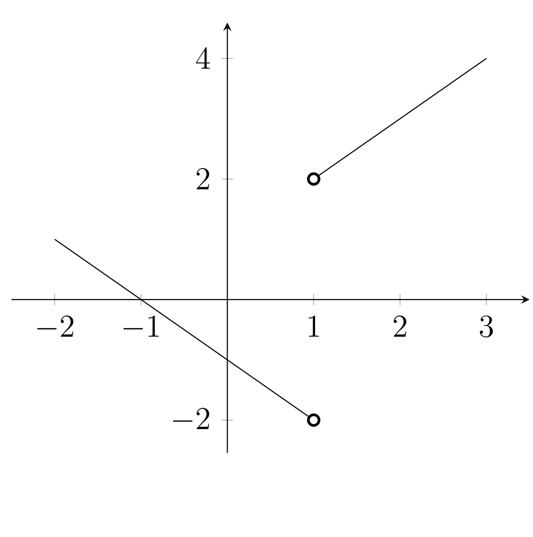
$$\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}=\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}\frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5}=\lim_{x\to -4}\frac{x^2+9-25}{(\sqrt{x^2+9}+5)(x+4)}$$
$$=\lim_{x\to -4}\frac{x^2-16}{(\sqrt{x^2+9}+5)(x+4)}=\lim_{x\to -4}\frac{(x+4)(x-4)}{(\sqrt{x^2+9}+5)(x+4)}$$
$$=\lim_{x\to -4}\frac{x-4}{\sqrt{x^2+9}+5}=\frac{-8}{10}=\frac{-4}{5}$$

2.3/38 Let $F(x)=\dfrac{x^2-1}{|x-1|}$. We have

$$\lim_{x\to 1^+}F(x)=\lim_{x\to 1^+}\frac{x^2-1}{x-1}=\lim_{x\to 1^+}x+1=2,$$

$$\lim_{x\to 1^-}F(x)=\lim_{x\to 1^-}\frac{x^2-1}{-(x-1)}=\lim_{x\to 1^-}-(x+1)=-2.$$

Hence $\lim\limits_{x\to 1}F(x)$ does not exist. Below is the graph of F



2.4/36 Find the values of a, b that make f continuous everywhere (there is a typo in the book, f should be defined at 2):

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & x < 2 \\ ax^2 - bx + 3 & 2 \le x < 3 \\ 2x - a + b & x \ge 3 \end{cases}$$

The only places where the function may not be continuous are $x = 2, 3$. At $x = 2$ we have

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^-} x + 2 = 4,$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax^2 - bx + 3 = 4a - 2b + 3.$$

For f to be continuous at $x = 2$ we must have equality of one-side limits with the function value, $4 = 4a - 2b + 3$.

At $x = 3$ we have

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} ax^2 - bx + 3 = 9a - 3b + 3,$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 2x - a + b = 6 - a + b.$$

For f to be continuous at $x = 3$ we must have equality of one-side limits with the function value, $9a - 3b + 3 = 6 - a + b$.

We get a system of linear equations

$$\begin{aligned} 4a - 2b &= 1 \\ 10a - 4b &= 3 \end{aligned}$$

which has the unique solution $a = 1/2, b = 1/2$.

2.5/16 We have

$$\lim_{x \to -3^-} \frac{x + 2}{x + 3} = +\infty$$

because the numerator goes to -1 as $x \to -3^-$ while the denominator goes to 0^- (goes to zero but stays negative).

2.5/27 We have

$$\begin{aligned} \lim_{x \to \infty} \left(\sqrt{9x^2 + x} - 3x \right) &= \lim_{x \to \infty} \left(\sqrt{9x^2 + x} - 3x \right) \frac{\sqrt{9x^2 + x} + 3x}{\sqrt{9x^2 + x} + 3x} \\ &= \lim_{x \to \infty} \frac{9x^2 + x - 9x^2}{\sqrt{9x^2 + x} + 3x} = \lim_{x \to \infty} \frac{x}{\sqrt{9x^2 + x} + 3x} \\ &= \lim_{x \to \infty} \frac{1}{\sqrt{9 + \frac{1}{x}} + 3} = \frac{1}{6}. \end{aligned}$$

2.5/36 We have

$$\lim_{x\to\pi/2^+} e^{\tan x} = \lim_{y\to-\infty} e^y = 0$$

since

$$\lim_{x\to\pi/2^+} \tan x = \lim_{x\to\pi/2^+} \frac{\sin x}{\cos x} = -\infty$$

(because $\sin x \to 1, \cos x \to 0^-$ as $x \to \pi/2^+$).

2.6/7 Find the equation of the tangent line to the curve

$$y = \sqrt{x}$$

through the point $(1, 1)$.

The slope of the tangent line is given by

$$\lim_{x\to1} \frac{\sqrt{x}-\sqrt{1}}{x-1} = \lim_{x\to1} \frac{\sqrt{x}-1}{x-1}\frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x\to1} \frac{x-1}{(x-1)(\sqrt{x}+1)}$$
$$= \lim_{x\to1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}.$$

Equivalently, the slope of the tangent line is

$$\lim_{h\to0} \frac{\sqrt{1+h}-\sqrt{1}}{h} = \lim_{h\to0} \frac{\sqrt{1+h}-1}{h}\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$
$$= \lim_{h\to0} \frac{1+h-1}{h(\sqrt{1+h}+1)} = \lim_{h\to0} \frac{h}{h(\sqrt{1+h}+1)}$$
$$= \lim_{h\to0} \frac{1}{\sqrt{1+h}+1} = \frac{1}{2}.$$

Hence the tangent line is given by

$$y - 1 = \frac{1}{2}(x-1).$$

2.6/8 Find the equation of the tangent line to the curve

$$y = \frac{2x+1}{x+2}$$

through the point $(1, 1)$.

The slope of the tangent line is given by

$$\lim_{x\to1} \frac{\frac{2x+1}{x+2}-\frac{3}{3}}{x-1} = \lim_{x\to1} \frac{(2x+1)-(x+2)}{(x+2)(x-1)}$$
$$= \lim_{x\to1} \frac{x-1}{(x+2)(x-1)} = \lim_{x\to1} \frac{1}{(x+2)} = \frac{1}{3}.$$

Equivalently, the slope of the tangent line is

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{2(1+h)+1}{(1+h)+2} - \frac{3}{3}}{h} &= \lim_{h\to 0} \frac{\frac{2h+3}{h+3} - 1}{h} = \lim_{h\to 0} \frac{2h+3-(h+3)}{h(h+3)} \\
&= \lim_{h\to 0} \frac{h}{h(h+3)} = \lim_{h\to 0} \frac{1}{(h+3)} = \frac{1}{3}.
\end{aligned}$$

Hence the tangent line is given by

$$y - 1 = \frac{1}{2}(x-1).$$