Quiz 1, MATH 1300-401

1.1/47 Find a function whose graph is the line segment between $(1, -3)$ and $(5, 7)$.

The line has slope
$$\frac{-3-7}{1-5} = \frac{5}{2}$$
so the line is given by $y - (-3) = \frac{5}{2}(x-1)$ or $y = \frac{5}{2}x - \frac{11}{2}$ as a function of x. Because we only want the segment joining the two points, we restrict the domain to $x \in [1, 5]$.

1.1/57 An open rectangular box of volume 2m^3 has a square base. Express the area of the box as a function of the length of a side of the base.

The volume V and the surface are S are given by
$$V(x,y) = 2 = x^2y, \; S(x,y) = x^2 + 4xy$$
where x is the length of the side of the base and y is the height of the box. Using the constraint $V(x,y) = 2$, we solve for y and substitute the expression into S to obtain a function of x
$$V(x,y) = 2 = x^2y, \; y = \frac{2}{x^2}, \; S(x,y) = x^2 + 4xy = x^2 + 4x\frac{2}{x^2} = x^2 + \frac{8}{x}.$$

1.2/9 Find an expression for a cubic function f if $f(1) = 6$, $f(-1) = f(0) = f(2) = 0$.

Since $f(x)$ has roots at $x = -1, 0, 2$ $f(x)$ must be divisible by $x(x+1)(x-2)$ (if a polynomial p has a root $p(c) = 0$, then $p(x) = q(x)(x-c)$ for some q). Since f is cubic, this determines f up to a constant, i.e.
$$f(x) = ax(x+1)(x-2)$$
for some constant a. We determine a using the information $f(1) = 6$:
$$f(1) = 6 = a(1)(1+1)(1-2), \; a = -3.$$
Hence
$$f(x) = -3x(x+1)(x-2) = -3x^3 + 3x^2 + 6x.$$

1.3/35 Find $f \circ g$, $g \circ f$, $f \circ f$, $g \circ g$ and their domains if
$$f(x) = x + \frac{1}{x}, \; g(x) = \frac{x+1}{x+2}.$$
We have (simplifying at the end)
$$f \circ g = g(x) + \frac{1}{g(x)} = \frac{x+1}{x+2} + \frac{1}{\frac{x+1}{x+2}} = \frac{2x^2+6x+5}{x^2+3x+2}, \text{ domain } x \neq -1, -2$$
$$g \circ f = \frac{f(x)+1}{f(x)+2} = \frac{x + \frac{1}{x} + 1}{x + \frac{1}{x} + 2} = \frac{x^2+x+1}{x^2+2x+1}, \text{ domain } x \neq -1, 0$$
$$f \circ f = f(x) + \frac{1}{f(x)} = x + \frac{1}{x} + \frac{1}{x + \frac{1}{x}} = \frac{x^4+3x^2+1}{x^3+x}, \text{ domain } x \neq 0$$
$$g \circ g = \frac{g(x)+1}{g(x)+2} = \frac{\frac{x+1}{x+2}+1}{\frac{x+1}{x+2}+2} = \frac{2x+3}{3x+5}, \text{ domain } x \neq -2, -5/3$$

1.6/26 Find the inverse of

$$y = \frac{e^x}{1+2e^x} = f(x).$$

Solving for x we have

$$\begin{aligned} y(1+2e^x) - e^x &= 0 \\ e^x(2y-1) &= -y \\ e^x &= \frac{y}{1-2y} \\ x &= \ln\left(\frac{y}{1-2y}\right). \end{aligned}$$

Hence

$$f^{-1}(x) = \ln\left(\frac{x}{1-2x}\right).$$

1.6/35 Find the exact values of $\log_5 125$, $\log_3 \frac{1}{27}$.

Since $5^3 = 125$ we have $\log_5 125 = 3$. We have

$$\log_3 \frac{1}{27} = -\log_3 27 = -3$$

since $3^3 = 27$ (or noting that $3^{-3} = 1/3^3 = 1/27$).

1.6/49 Solve the following for x

$$e^{7-4x} = 6, \ \ln(3x-10) = 2.$$

We have

$$\begin{aligned} 7 - 4x &= \ln 6 \\ x &= \frac{1}{4}(7 - \ln 6) \end{aligned}$$

and

$$\begin{aligned} 3x - 10 &= e^2 \\ x &= \frac{1}{3}(e^2 + 10). \end{aligned}$$

1.6/50 Solve the following for x

$$\ln(x^2 - 1) = 3, \ e^{2x} - 3e^x + 2 = 0.$$

We have

$$\begin{aligned} x^2 - 1 &= e^3 \\ x &= \pm\sqrt{e^3+1} \end{aligned}$$

and factoring the second equation (it's quadratic in e^x, $e^{2x} = (e^x)^2$)

$$\begin{aligned}(e^x - 2)(e^x - 1) &= 0\\ e^x &= 2, 1\\ x &= \ln 2,\ 0.\end{aligned}$$

Apx C/30 Find all values of x in the interval $[0, 2\pi]$ that satitisfy $2\sin^2 x = 1$.

In other words, $\sin x = \pm 1/\sqrt{2}$. The values of x are given by $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ (draw a picture and remember that $\sin(\pi/4) = 1/\sqrt{2}$).

Apx C/41 Prove the **law of cosines**

$$a^2 + b^2 = c^2 - 2ab\cos(\theta)$$

where a, b, c are the (lengths of the) sides of a triangle and θ is the angle between sides a, b.

In the picture below, we note that the coordinates of the point at the top of the triangle are $(b\cos\theta, b\sin\theta)$ so that the length of c is determined by

$$c^2 = (a - b\cos\theta)^2 + (0 - b\sin\theta)^2 = a^2 - 2ab\cos\theta + b^2(\cos^2\theta + \sin^2\theta) = a^2 + b^2 - 2ab\cos\theta.$$