Math 1300-001, Quiz 9 Name: ______________________

This quiz is due in class Monday, April 20th.

1. Find f, g, h if

   (a) $f'(x) = \dfrac{4}{\sqrt{1-x^2}},\ f(1/2) = 0$

$$\begin{aligned} f(x) &= 4\arcsin(x) + C \\ 0 &= f(1/2) = 4\arcsin(1/2) + C = 4\pi/6 + C \Rightarrow C = -2\pi/3 \\ f(x) &= 4\arcsin(x) - 2\pi/3. \end{aligned}$$

   (b) $g''(x) = \dfrac{3}{\sqrt{x}},\ g(4) = 20,\ g'(4) = 7$

$$\begin{aligned} g'(x) &= 3x^{1/2}/(1/2) + C \\ 7 &= g'(4) = 6(4)^{1/2} + C \Rightarrow C = -5 \\ g(x) &= 6x^{3/2}/(3/2) - 5x + D \\ 20 &= g(4) = 4(4)^{3/2} - 5(4) + D \Rightarrow D = 8 \\ g(x) &= 4x^{3/2} - 5x + 8. \end{aligned}$$

   (c) $h''(x) = 2e^x + 3\sin x,\ h(0) = 0 = h(\pi)$

$$\begin{aligned} h'(x) &= 2e^x - 3\cos(x) + C \\ h(x) &= 2e^x - 3\sin(x) + Cx + D \\ 0 &= h(0) = 2e^0 - 3\sin(0) + C(0) + D \Rightarrow D = -2 \\ 0 &= h(\pi) = 2e^\pi - 3\sin(\pi) + C(\pi) - 2 \Rightarrow C = 2(1-e^\pi)/\pi \\ h(x) &= 2e^x - 3\sin(x) + \frac{2(1-e^\pi)}{\pi}x - 2 \end{aligned}$$

2. Consider the function $f(x) = x(x-1)(x+1)$ on the interval $[-1, 0]$.

(a) Find the exact area under the curve $y = f(x)$ and above the x-axis

$$\text{Area } = \int_{-1}^{0} f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

by taking a limit of Riemann sums. You may need the formulae

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \ \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \ \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

We have

$$f(x) = x^3 - x, \ x_i = -1 + i\Delta x, \ \Delta x = \frac{1}{n}.$$

The limit we want to find is

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 - x_i^2)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(-1 + \frac{i}{n}\right)^3 - \left(-1 + \frac{i}{n}\right)\right)\frac{1}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[(-1)^3 + 3(-1)^2\left(\frac{i}{n}\right) + 3(-1)\left(\frac{i}{n}\right)^2 + \left(\frac{i}{n}\right)^3 + 1 - \frac{i}{n}\right]\frac{1}{n} \\
&= \lim_{n\to\infty} \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{3}{n^3}\sum_{i=1}^{n} i^2 + \frac{1}{n^4}\sum_{i=1}^{n} i^3 \\
&= \lim_{n\to\infty} \frac{2}{n^2}\frac{n(n+1)}{2} - \frac{3}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2 \\
&= 1 - 1 + \frac{1}{4} = \frac{1}{4}.
\end{aligned}$$

(b) Find the exact area under the curve $y = f(x)$ and above the x-axis using the fundamental theorem of calculus

$$\int_a^b f(t)dt = F(b) - F(a) \text{ where } F'(x) = f(x).$$

We have

$$\int_{-1}^{0} f(x)dx = \int_{-1}^{0} (x^3 - x)dx = \frac{x^4}{4} - \frac{x^2}{2}\Big|_{-1}^{0} = 0 - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{4}.$$