Math 1300-001, Quiz 3 Name: ________________________

This is a take-home quiz, due Monday, February 16th at the beginning of class. Please use this as a cover page for your work.

1. Using the definition of the derivative as limit of difference quotients, find the derivatives of the following. Clearly show all of your work.

   (a) $y = x^4$ (use binomial theorem, $(x+h)^4 = ?$)
   To simplify the numerator below, we note that

$$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$$

   by the binomial theorem (fourth row of Pascal's triangle).
   We have

$$\begin{aligned} y' &= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} = \lim_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\ &= \lim_{h\to 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3)}{h} = \lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2 + h^3 \\ &= 4x^3. \end{aligned}$$

   (b) $y = x^{1/3}$ (use difference of cubes, $((x+h)^{1/3})^3 - (x^{1/3})^3 = ?$)
   To simplify the numerateor below, we use the "difference of cubes" factorization

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

   with $a = (x+h)^{1/3}, b = x^{1/3}$.
   We have

$$\begin{aligned} y' &= \lim_{h\to 0} \frac{(x+h)^{1/3} - x^{1/3}}{h} = \lim_{h\to 0} \frac{((x+h)^{1/3} - x^{1/3})((x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3})}{h((x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3})} \\ &= \lim_{h\to 0} \frac{(x+h) - x}{h((x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3})} = \lim_{h\to 0} \frac{1}{(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}} \\ &= \frac{1}{(x+0)^{2/3} + (x+0)^{1/3}x^{1/3} + x^{2/3}} = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}. \end{aligned}$$

2. Find the equations of the tangent lines to the graph of $y = x^2$ that also go through the point $(0, -4)$. (Draw a picture; there are two such lines.)

   The slope of the line going through (x, x^2) and $(0, -4)$ is

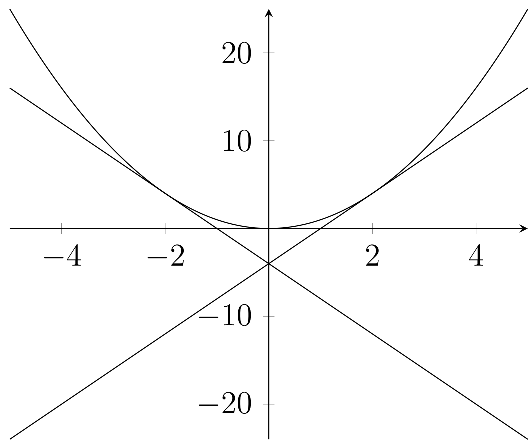
$$m = \frac{x^2 - (-4)}{x - 0} = x + \frac{4}{x}.$$

   This should be the derivative of $y = x^2$ at x. So we have

$$y'(x) = 2x = x + \frac{4}{x} \Leftrightarrow x^2 - 4 = 0 \Leftrightarrow x = \pm 2, m = \pm 4.$$

   The equations for these lines are

$$y - 4 = \pm 4(x - \pm 2) \text{ or } y = \pm 4x - 4.$$



3. Let $f(x) = x^4 - 18x^2 + 77$. Find exact values in your answers to the folloing questions, i.e. don't use decimal approximations.

   (a) For what values of x is $f(x) = 0$? (If $y = x^2$ then $f(y) = y^2 - 18y + 77$. Use the quadratic formula to find y. You should get four values for x).
   If $y^2 - 18y + 77 = 0$, then $y = \frac{18 \pm \sqrt{18^2 - 4 \cdot 77}}{2} = 11, 7$, and if $y = x^2$ then $x = \pm\sqrt{11} \approx \pm 3.317, \pm\sqrt{7} \approx \pm 2.646$.

   (b) Find $f'(x)$ and the values of x for which $f'(x) = 0$
   $f'(x) = 4x^3 - 36x = 4x(x-3)(x+3) = 0$ when $x = 0, \pm 3$.

   (c) On what intervals is f increasing? Decreasing?
   f' is positive (and f is increasing) on $(-3, 0) \cup (3, \infty)$.
   f' is negative (and f is decreasing) on $(-\infty, -3) \cup (0, 3)$.

   (d) Find the points on the graph of f where f has a local maximum or local minimum.
   There is a local maximum of $f(0) = 77$ when $x = 0$ and local minima of $f(\pm 3) = -4$ at $x = \pm 3$.

   (e) Find $f''(x)$ and the values of x for which $f''(x) = 0$.
   $f''(x) = 12x^2 - 36 = 12(x - \sqrt{3})(x + \sqrt{3}) = 0$ when $x = \pm\sqrt{3}$.

   (f) On what intervals is the graph of f concave up? Concave down?
   $f'' > 0$ (and the graph of f is concave up) on $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$.
   $f'' < 0$ (and the graph of f is concave down) on $(-\sqrt{3}, \sqrt{3})$.

   (g) Find the coordinates of any inflection points.
   The inflection points are $(\pm\sqrt{3}, 32)$

   (h) Use the above information to sketch a graph of f.