Math 1300-001, Quiz 2 Name: ________________________

1. Consider the rational function

$$g(x) = \frac{2x^2 - 2}{x^2 - 3x + 2}.$$

(a) For what values of x is g discontinuous?

The rational function g is continuous on its domain i.e. away from the zeros of the denominator $x^2 - 3x + 2 = (x-2)(x-1)$, $x = 1, 2$.

(b) Find the left/right limits (possibly $\pm\infty$) of g at each of the x-values from (a).

The discontinuity at $x = 1$ is removable since

$$g(x) = \frac{2x^2 - 2}{x^2 - 3x + 2} = \frac{2(x-1)(x+1)}{(x-2)(x-1)} = \frac{2(x+1)}{(x-2)} \text{ if } x \neq 1.$$

So we have

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} \frac{2(x+1)}{(x-2)} = \frac{2(1+1)}{1-2} = -4.$$

(in particular $\lim\limits_{x \to 1^-} g(x) = \lim\limits_{x \to 1^+} g(x)) = -4$).

The discontinuity at $x = 2$ is an infinite discontinuity, i.e. the line $x = 2$ is a vertical asymptote.

As $x \to 2^+$, the numerator $2x^2 - x$ of g approaches 6 while the denominator $x^2 - 3x + 2 = (x-2)(x-1)$ of g approaches 0 *from the right*; the top goes to 6 while the bottom goes to 0 but *stays positive*. Hence $\lim_{x \to 2^+} g(x) = +\infty$.

As $x \to 2^-$, the numerator $2x^2 - x$ of g approaches 6 while the denominator $x^2 - 3x + 2 = (x-2)(x-1)$ of g approaches 0 *from the left*; the top goes to 6 while the bottom goes to 0 but *stays negative*. Hence $\lim_{x \to 2^+} g(x) = -\infty$.

(c) What is $\lim\limits_{x \to \infty} g(x)$?

We have

$$\begin{aligned}
\lim_{x \to \infty} g(x) &= \lim_{x \to \infty} \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \frac{2x^2 - 2}{x^2 - 3x + 2} \\
&= \lim_{x \to \infty} \frac{2 - \frac{2}{x^2}}{1 - \frac{3}{x} + \frac{2}{x^2}} = \frac{2 - 2 \lim\limits_{x \to \infty} \dfrac{1}{x^2}}{1 - 3 \lim\limits_{x \to \infty} \dfrac{1}{x} + 2 \lim\limits_{x \to \infty} \dfrac{1}{x^2}} \\
&= \frac{2 - 2 \cdot 0}{1 - 3 \cdot 0 + 2 \cdot 0} = 2.
\end{aligned}$$

2. Consider the following function (where $a, b \in \mathbb{R}$ are constants)

$$f(x) = \begin{cases} 3x - 2 & \text{if } x < -2 \\ ax^2 + bx + 1 & \text{if } -2 \leq x < 3 \\ ax + b & \text{if } x \geq 3 \end{cases}$$

(a) Determine $\lim_{x \to -2^-} f(x)$, $\lim_{x \to -2^+} f(x)$, $\lim_{x \to 3^-} f(x)$, and $\lim_{x \to 3^+} f(x)$.

We have

$$\begin{aligned} \lim_{x \to -2^-} f(x) &= \lim_{x \to -2^-} 3x - 2 = -8 \\ \lim_{x \to -2^+} f(x) &= \lim_{x \to -2^+} ax^2 + bx + 1 = 4a - 2b + 1, \end{aligned}$$

and

$$\begin{aligned} \lim_{x \to 3^-} f(x) &= \lim_{x \to 3^-} ax^2 + bx + 1 = 9a + 3b + 1 \\ \lim_{x \to 3^+} f(x) &= \lim_{x \to 3^+} ax + b = 3a + b. \end{aligned}$$

(b) Find values of a and b that will make f continuous on $(-\infty, \infty)$. [Write down two equations in the two unknowns a and b, one to describe continuity at $x = -2$ and another to describe continuity at $x = 3$.]

The only possible discontinuities of f are at $x = -2, 3$. For f to be continuous at $x = -2$ we must have

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} f(x) = f(-2), \text{ i.e. } -8 = 4a - 2b + 1.$$

For f to be continuous at $x = 3$ we must have

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} f(x) = f(3), \text{ i.e. } 9a + 3b + 1 = 3a + b.$$

For both to hold, we need to solve the system of linear equations

$$\begin{aligned} 4a - 2b &= -9 \\ 6a + 2b &= -1. \end{aligned}$$

Adding the two equations together we get $10a = -10$ and $a = -1$. Substituting $a = -1$ into either equation gives $b = 5/2$. So for f to be continuous everywhere, we must have $a = -1, b = 5/2$.