Math 1300-001, Quiz 1 Name: ______________________

The two problems are very similar, finding limits of "difference quotients" for the tangent problem in 1) and the velocity problem in 2). As we will see, we are finding derivatives, $f'(1)$ in 1) and $h'(1)$ in 2).

1. (a) Write an expression for the slope $m(t)$ of the secant line through the points $(1, 1/3)$ and $(t, f(t))$ on the graph of $f(x) = \frac{x}{2+x}$.

   We take the difference in the second coordinates divided by the difference in the first coordinates of the two points $(t, f(t)), (1, 1/3)$ to get

$$m(t) = \frac{\frac{t}{2+t} - \frac{1}{3}}{t-1}.$$

   (b) Simplify the resulting expression to find the slope of the tangent line to the graph of $f(x)$ going through the point $(1, 1/3)$. In other words, find $\lim_{t\to 1} m(t)$.

   We have

$$\begin{aligned}\lim_{t\to 1} m(t) &= \lim_{t\to 1} \frac{\frac{t}{2+t} - \frac{1}{3}}{t-1} \\ &= \lim_{t\to 1} \frac{\frac{3t-(2+t)}{3(2+t)}}{t-1} = \lim_{t\to 1} \frac{2(t-1)}{3(2+t)(t-1)} \\ &= \lim_{t\to 1} \frac{2}{3(2+t)} = \frac{2}{3(2+1)} = \frac{2}{9}.\end{aligned}$$

   (c) Write an equation for the tangent line to the graph $f(x)$ through the point $(1, 1/3)$ (you know the slope of the line and a point on the line).

   We want the equation for a line with slope $1/3$ going through the point $(1, 1/3)$. This is

$$y - \frac{1}{3} = \frac{2}{9}(x-1), \text{ or } y = \frac{2}{9}x + \frac{1}{9}.$$

2. A height (in feet after t seconds) of a ball thrown straight into the air from 8 ft with an initial velocity of 8 ft/s is given by

$$h(t) = -16t^2 + 8t + 8.$$

(a) At what time does the ball hit the ground? (For what $t_0 > 0$ is $h(t_0) = 0$?) The answer is $t_0 = 1$.

Factoring, we have

$$h(t) = -16t^2 + 8t + 8 = -8(2t + 1)(t - 1)$$

with zeros $t = -1/2, 1$. We are looking for the positive solution, $t_0 = 1$ second.

Or we can use the quadratic fomula

$$h(t) = 0 \Leftarrow t = \frac{-8 \pm \sqrt{8^2 - 4(-16)8}}{2(-16)} = \frac{8 \pm 24}{-32} = -1/2, 1$$

where we take the positive solution, $t_0 = 1$.

(b) Find an expression for the average velocity $v(x)$ of the ball over the time interval $[x, t_0]$.

The average velocity over $[x, 1]$ is

$$v(x) = \frac{h(1) - h(x)}{1 - x} = \frac{0 - (-16x^2 + 8x + 8)}{1 - x} = \frac{-16x^2 + 8x + 8}{x - 1}.$$

(c) Find the instantaneous velocity of the ball when it hits the ground, i.e. find $\lim_{x \to t_0} v(x)$.

The instantaneous velocity at $t_0 = 1$ is

$$\begin{aligned} \lim_{xa \to 1} v(x) &= \lim_{x \to 1} \frac{-16x^2 + 8x + 8}{x - 1} \\ &= \lim_{x \to 1} \frac{-8(2x + 1)(x - 1)}{x - 1} = \lim_{x \to 1} -8(2x + 1) \\ &= -8(2(1) + 1) = -24 \text{ ft/s}. \end{aligned}$$