Math 1300-001, Review DOUBLE Quiz 11/12   Name: ______________________

This quiz is due Friday, May 1st in class.

1. Find the following limits

   (a) $\lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2}$

$$\begin{aligned}\lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2} &= \lim_{x\to 3} \frac{(\sqrt{x+6}-x)(\sqrt{x+6}+x)}{(x^3-3x^2)(\sqrt{x+6}+x)} = \lim_{x\to 3} \frac{(x+6)-x^2}{(x^2(x-3)(\sqrt{x+6}+x)} \\ &= \lim_{x\to 3} \frac{-(x-3)(x+2)}{(x^2(x-3)(\sqrt{x+6}+x)} = \lim_{x\to 3} \frac{-(x+2)}{(x^2(\sqrt{x+6}+x)} = -\frac{5}{54}.\end{aligned}$$

   (b) $\lim_{x\to\pi^-} \ln(\sin x)$

   As $x\to\pi^-$, $\sin x \to 0^+$, so

$$\lim_{x\to\pi^-} \ln(\sin x) = \lim_{y\to 0^+} \ln y = -\infty.$$

   (c) $\lim_{x\to 0} x^2\cos(x^{-2})$ (Hint: "squeeze" theorem)

   We have $-1 \le \cos x \le 1$ for any x, so that

$$-x^2 \le x^2\cos(x^{-2}) \le x^2.$$

   As $x\to 0$, $\pm x^2 \to 0$, so that $x^2\cos(x^{-2}) \to 0$ as well.

2. Use the intermediate value theorem to show that $x = e^{-x^2}$ has a solution.

   The intermediate value theorem states that for a continuous function f defined on a closed interval $[a,b]$ and for any number N between $f(a)$ and $f(b)$, there will be a $c\in[a,b]$ such that $f(c)=N$. If we consider the continuous funtion $f(x) = x - e^{-x^2}$ on the interval $[0,1]$ for instance, we see that $f(0) = -1 < 0, f(1) = 1-1/e > 0$ so that $N=0$ is between $f(0)$ and $f(1)$. Hence, for some number $c\in(0,1)$, $f(c)=0$.

3. Use the definition of the derivative (as a limit of averages) to find $\frac{d}{dx}\sqrt{1+2x}$. Find the tangent line to the graph of $y=\sqrt{1+2x}$ when $x=4$.

$$\begin{aligned}\frac{d}{dx}\sqrt{1+2x} &= \lim_{h\to 0} \frac{\sqrt{1+2(x+h)}-\sqrt{1+2x}}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{1+2(x+h)}-\sqrt{1+2x})(\sqrt{1+2(x+h)}+\sqrt{1+2x})}{h(\sqrt{1+2(x+h)}+\sqrt{1+2x})} \\ &= \lim_{h\to 0} \frac{(1+2(x+h))-(1+2x)}{h(\sqrt{1+2(x+h)}+\sqrt{1+2x})} = \lim_{h\to 0} \frac{2h}{h(\sqrt{1+2(x+h)}+\sqrt{1+2x})} \\ &= \lim_{h\to 0} \frac{2}{(\sqrt{1+2(x+h)}+\sqrt{1+2x})} = \frac{1}{\sqrt{1+2x}}.\end{aligned}$$

For $f(x)=\sqrt{1+2x}$, we have $f(4)=3, f'(4)=1/3$ so that the tangent line is

$$y-f(4)=f'(4)(x-4),\ \ y-3=(x-4)/3.$$

4. Use the definition of the derivative (as a limit of averages) to find $\dfrac{d}{dx}\dfrac{1}{4x+1}$. Find the tangent to the graph of $y=1/(4x+1)$ when $x=4$.

$$\begin{aligned}\frac{d}{dx}\frac{1}{4x+1} &= \lim_{h\to 0}\frac{\frac{1}{4(x+h)+1}-\frac{1}{4x+1}}{h}\\ &= \lim_{h\to 0}\frac{4x+1-4(x+h)-1}{(4(x+h)+1)(4x+1)h}\\ &= \lim_{h\to 0}\frac{4h}{(4(x+h)+1)(4x+1)h}\\ &= \lim_{h\to 0}\frac{4}{(4(x+h)+1)(4x+1)}=\frac{4}{(4x+1)^2}.\end{aligned}$$

For $f(x)=1/(4x+1)$, we have $f(4)=1/17, f'(4)=4/289$ so that the tangent line is

$$y-f(4)=f'(4)(x-4),\ \ y-1/17=4(x-4)/289.$$

5. Differentiate the following functions

(a) $\dfrac{e^{1/x}}{x^2}$

$$\frac{d}{dx}\frac{e^{1/x}}{x^2}=\frac{x^2(-x^{-2}e^{1/x})-e^{1/x}2x}{(x^2)^2}.$$

(b) $\dfrac{\sec(2\theta)}{1+\tan(2\theta)}$

$$\frac{d}{dx}\frac{\sec(2\theta)}{1+\tan(2\theta)}=\frac{[1+\tan(2\theta)]2\sec(2\theta)\tan(2\theta)-2\sec(2\theta)\sec^2(2\theta)}{(1+\tan(2\theta))^2}.$$

(c) $3^{x\ln x}$

$$\frac{d}{dx}3^{x\ln x}=3^{x\ln x}(\ln 3)(x\frac{1}{x}+\ln x).$$

(d) $\arctan(\arcsin(\sqrt{x}))$

$$\frac{d}{dx}\arctan(\arcsin(\sqrt{x}))=\frac{1}{1+(\arcsin(\sqrt{x}))^2}\frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{1}{2\sqrt{x}}.$$

6. Find dy/dx if x and y are related by

$$x^2 \cos y + \sin(2y) = xy.$$

Differenting the equation with respect to x gives

$$x^2(-y' \sin y) + 2x \cos(2y) + 2y' \cos y = xy' + y.$$

Solving for y' we get

$$y' = \frac{y - 2x\cos(2y)}{2\cos y - x^2 \sin y - x}.$$

7. Use a tangent line approximation (to $\sqrt[3]{x}$ for instance) to estimate $\sqrt[3]{9}$.

The tangent line approximation to $f(x)$ at a is given by

$$f(x) \approx f(a) + f'(a)(x - a).$$

If $f(x) = \sqrt[3]{x}$ and we look near $a = 8$ then

$$f(9) \approx 2 + \frac{1}{12}(9 - 8) = \frac{25}{12} = 2.08\bar{3}$$

since $\sqrt[3]{8} = 2$ and $f'(x) = x^{-2/3}/3, f'(8) = 1/12$. The actually value is closer to $2.080083\ldots$

8. Use l'Hôpital's rule to find the following limits

(a) $\lim_{x\to\frac{\pi}{2}^-} (\tan x)^{\cos x}$

This is indeterminate, of the form ∞^0. We consider the logarithm of the function in question

$$\lim_{x\to\frac{\pi}{2}^-} \cos x \ln(\tan x) = \lim_{x\to\frac{\pi}{2}^-} \frac{\ln(\tan x)}{\sec x}$$

which now looks like ∞/∞. Apply l'Hôpital's rule

$$\lim_{x\to\frac{\pi}{2}^-} \ln\left((\tan x)^{\cos x}\right) \overset{\text{l'H}}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{\sec^2 x/\tan x}{\sec x \tan x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\sec x}{\tan^2 x}$$

which is still indeterminate of the form ∞/∞. We apply l'Hôpital's rule again to get

$$\lim_{x\to\frac{\pi}{2}^-} \ln\left((\tan x)^{\cos x}\right) \overset{\text{l'H}}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{\sec x \tan x}{2\tan x \sec^2 x} = \lim_{x\to\frac{\pi}{2}^-} \frac{1}{2}\cos x = 0.$$

Or, to avoid using l'Hôpital's rule again, simplify

$$\frac{\sec x}{\tan^2 x} = \frac{\cos^2 x}{\cos x \sin^2 x} = \frac{\cos x}{\sin^2 x} = 0.$$

We exponentiate to get

$$\lim_{x\to\frac{\pi}{2}^-} (\tan x)^{\cos x} = \lim_{x\to\frac{\pi}{2}^-} e^{\ln((\tan x)^{\cos x})} = e^{\left(\lim_{x\to\frac{\pi}{2}^-} \ln\left((\tan x)^{\cos x}\right)\right)} = e^0 = 1.$$

(b) $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$ $(a, b$ constants)

This is indeterminate of the form 1^∞. We consider the logarithm of the function in question

$$\lim_{x\to\infty} bx\ln\left(1+\frac{a}{x}\right) = \lim_{x\to\infty}\frac{b\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}}$$

which is indeterminate of the form $0/0$ so that we can apply l'Hôpital's rule. We get

$$\lim_{x\to\infty}\ln\left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty}\frac{b\,(1+a/x)^{-1}\,(-ax^{-2})}{-x^{-2}} = \lim_{x\to\infty}\frac{ab}{1+a/x} = ab.$$

We exponentiate to get

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = e^{ab}.$$

9. Find the dimensions (radius, height) of a cylindrical can with minimal surface area, if it has no top and its volume is 500 cm^3.

The surface of the can consists of the bottom and the "side"

$$S = \pi r^2 + 2\pi rh$$

in terms of the radius and height. The radius and height are not independent; the volume constraint gives

$$500 = \pi r^2 h,\ \ h = \frac{500}{\pi r^2}.$$

Hence the surface area, as a function of the radius, is give by

$$S(r) = \pi r^2 + \frac{1000}{r},\ \ r \in (0\infty).$$

We look for critical numbers (r such that $S'(r) = 0$)

$$S'(r) = 2\pi r - \frac{1000}{r^2} = 0,\ \ r = (500/\pi)^{1/3} =: r_0.$$

This is a minimum since $S' < 0$ for $r \in (0, r_0)$ and $S'(r) > 0$ for $r \in (r_0, \infty)$. So the dimensions of the can with minimal surface area are

$$r_0 = (500/\pi)^{1/3}\ h_0 = \frac{500}{\pi r_0^2} = (500/\pi)^{1/3}$$

10. Water is leaking out of an inverted conical tank at a rate of $10,000\ \mathrm{cm}^3/\mathrm{min}$ at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which the water is being pumped into the tank.

The change in volume is

$$\frac{dV}{dt} = R - 10000\ \mathrm{cm}^3/\mathrm{min}$$

where R is the rate of water being pumped into the tank. We are also given the information

$$\left.\frac{dh}{dt}\right|_{h=2\ \mathrm{m}} = 20\ \mathrm{cm}/\mathrm{min}$$

where h is the height of the water in the tank.

The volume, height, and radius (of the water in the tank) are related by

$$V = \frac{1}{3}\pi r^2 h.$$

But r and h are not independent; using similar triangles we get

$$\frac{2}{6} = \frac{r}{h},\quad r = \frac{h}{3},$$

so that the volume and height are related by

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (h/3)^2 h = \pi h^3/27.$$

Differentiate to relate the rates dV/dt and dh/dt

$$\frac{dV}{dt} = \frac{\pi}{9} h^2 \frac{dh}{dt}.$$

Now use the specific information about dh/dt (and put everything in centimeters)

$$R - 10000 = \left.\frac{dV}{dt}\right|_{h=2\ \mathrm{m}} = \frac{\pi}{9}(200)^2 20 = 800000\pi/9,$$

$$R = 10000 + 800000\pi/9\ \mathrm{cm}^3/\mathrm{min}.$$

11. Consider the function

$$f(x) = \frac{x^2 - 16}{x - 5}$$

(a) What is the domain of f? For what values of x is $f(x) = 0$?

(b) Find f' and list the intervals on which f is increasing/decreasing.

(c) Find f'' and list the intervals on which f is concave up/concave down.

(d) List and classify any local extrema for the function f. Does the graph of f have any inflection points?

(e) Using the information above, sketch the graph of f.

We have

$$f'(x) = \frac{(x-5)(2x) - (x^2 - 16)}{(x-5)^2} = \frac{x^2 - 10x + 16}{(x-5)^2} = \frac{(x-8)(x-2)}{(x-5)^2},$$

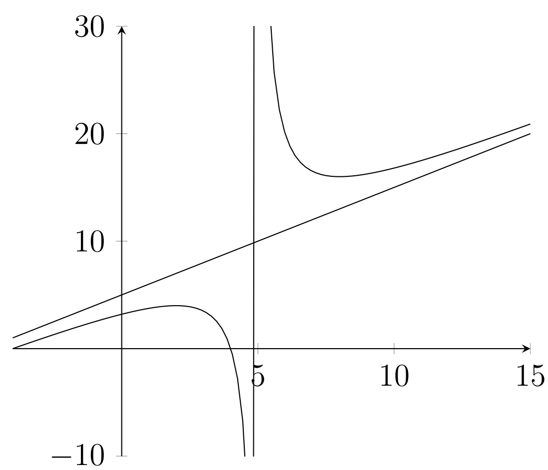
$$f''(x) = \frac{(x-5)^2(2x-10) - 2(x-5)(x^2 - 10x + 16)}{(x-5)^4} = \frac{18}{(x-5)^3}.$$

Hence f is increasing on $(-\infty, 2) \cup (8, \infty)$, decreasing on $(2, 5) \cup (5, 8)$ with a local max of $f(2) = 4$ at $x = 2$ and a local min of $f(8) = 16$ at $x = 8$. The graph of f is concave up on $(5, \infty)$ and concave down on $(-\infty, 5)$, with no inflection points (the function is undefined at $x = 5$ where it has a vertical asymptote).

Some other information to note is that

$$f(\pm 4) = 0, \lim_{x \to \infty} f(x) = \infty, \lim_{x \to -\infty} f(x) = -\infty, \lim_{x \to 5^-} f(x) = -\infty, \lim_{x \to 5^+} f(x) = \infty.$$

The graph looks like (vertical asymptote $x = 5$ and slant asymptote $y = x + 5$ shown):



12. Subdivide the interval $[-3, 5]$ into four equal parts and use a left endpoint Riemann sum to estimate the definite integral

$$\int_{-3}^{5} (x^2 - 1)dx.$$

We have $\Delta x = (5 - -3)/4 = 2, x_i = -3 + i\Delta x = -3 + 2i, f(x_i) = (-3 + 2i)^2 - 1$. Hence the desired approximation is

$$\sum_{i=0}^{3} f(x_i)\Delta x = 2(8 + 0 + 0 + 8) = 32.$$

13. Compute the following definite integrals

(a) $\displaystyle\int_{1/2}^{3/2} x\sqrt{2x - 1}dx$

Let $u = 2x - 1, x = (u + 1)/2, du = 2dx, u(1/2) = 0, u(3/2) = 2$, so that the integral becomes

$$\frac{1}{4}\int_{0}^{2} (u + 1)\sqrt{u}dx = \frac{u^{5/2}}{10} + \frac{u^{3/2}}{6}\Bigg|_0^2 = \frac{2^{3/2}}{5} + \frac{2^{1/2}}{3}.$$

(b) $\displaystyle\int_{0}^{\pi/2} \cos x \sin(\sin x)dx$

Let $u = \sin x, du = \cos x dx, u(0) = 0, u(\pi/2) = 1$, so that the integral becomes

$$\int_{0}^{1} \sin(u)du = -\cos x\Big|_0^1 = 1 - \cos 1.$$

(c) $\displaystyle\int_{1}^{e} \frac{dx}{x\sqrt{\ln x}}$

Let $u = \ln x, du = dx/x, u(1) = 0, u(e) = 1$, so that the integral becomes

$$\int_{0}^{1} \frac{du}{\sqrt{u}} = 2u^{1/2}\Big|_0^1 = 2.$$

14. Find the area bounded by the curves

$$y = 2 - x,\ y = x^2.$$

The points of intersection are solutions of $x^2 = 2 - x$, i.e. $x = -2, 1$, and $2 - x \geq x^2$ on $[-2, 1]$ so the area is "$\int$ top − bottom"

$$\int_{-2}^{1} (2 - x - x^2)dx = 2x - x^2/2 - x^3/3\Big|_{-2}^{1} = (2 - 1/2 - 1/3) - (-4 - 2 + 8/3) = 9/2.$$