Math 1300-018 Quiz 9 Name:________________________

1. Find the following limits if they exist. MAKE SURE L'HOSPITAL'S RULE APPLIES BEFORE USING IT.

   (a)
$$\lim_{x\to\infty} x^{1/x}, \ \lim_{x\to 0^+} x^{1/x}$$

   First note that
$$\lim_{x\to 0^+} x^{1/x} = \text{“}0^{\infty}\text{”} = 0$$
   is not indeterminate.
   However, $\lim_{x\to\infty} x^{1/x} = \text{“}\infty^{\infty}\text{”}$ is. We take logarithms and consdier $\lim_{x\to\infty} \frac{\ln x}{x} = \text{“}-\infty/\infty\text{”}$ where L'Hospital's rule applies. Differentiating, we get
$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$$
   so that
$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln(x^{1/x})} = e^{\ln(\lim_{x\to\infty} \frac{\ln x}{x})} = e^0 = 1.$$

   (b)
$$\lim_{x\to\infty} x\tan(1/x), \ \lim_{x\to 2/\pi^+} x\tan(1/x)$$

   First note that
$$\lim_{x\to 2/\pi^+} x\tan(1/x) = (\pi/2)(\infty) = \infty$$
   dosn't require L'Hospital's rule. To see the ∞ above, note that if $x > 2/\pi$, then $1/x < \pi/2$ so that
$$\lim_{x} \to 2/\pi^+ \tan(1/x) = \lim_{y\to\pi/2^-} \tan(y) = +\infty.$$
   However $\lim_{x\to\infty} x\tan(1/x) = \text{“}\infty\cdot 0\text{”}$ is indeterminate. We rewrite this product as a quotient and consider $\lim_{x\to\infty} \tan(1/x)/(1/x) = \text{“}0/0\text{”}$ where L'Hospital's rule applies. Differentiating the numerator and denomiator gives
$$\lim_{x\to\infty} x\tan(1/x) = \lim_{x\to\infty} \frac{\tan(1/x)}{1/x} = \lim_{x\to\infty} \frac{\sec^2(1/x)(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \sec^2(1/x) = 1.$$

   (c)
$$\lim_{x\to 0^+} (\sin x)^{\sin x}, \ \lim_{x\to\pi/2} \sin x^{\sin x}$$

   First note that
$$\lim_{x\to\pi/2} \sin x^{\sin x} = 1^1 = 1$$
   doesn't require L'Hospital's rule.

However $\lim_{x\to 0^+}(\sin x)^{\sin x} = $ “0^0” is indeterminate. We take logarithms and consider

$$\lim_{x\to 0^+} \sin x \ln(\sin x) = 0(-\infty).$$

We rewrite this as

$$\lim_{x\to 0^+} \ln(\sin x)/(\csc x) = \text{“}-\infty/\infty\text{”}$$

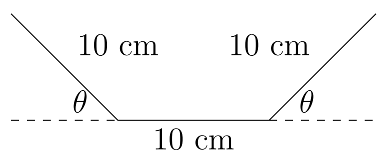
where L’Hospital’s rule applies. We the consider the limit of the quotient of the derivatives

$$\lim_{x\to 0^+} \frac{\ln(\sin x)}{\csc x} = \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{-\csc x \cot x} = \lim_{x\to 0^+} -\sin x = 0.$$

The is the limit of the logarithm of what we wanted, so we exponentiate

$$\lim_{x\to 0^+} (\sin x)^{\sin x} = \lim_{x\to 0^+} e^{\ln((\sin x)^{\sin x})} = e^{\lim_{x\to 0^+} \ln((\sin x)^{\sin x})} = e^0 = 1.$$

2. A long sheet of metal of width 30 cm will be bent into a gutter as shown. Find θ such that the gutter will have maximum capacity.



(Solution 1) We want to maximize the area of the (trapezoidal) cross section, $A = h(b_1+b_2)/2$ where h is the distance between the parallel sides and b_1 and b_2 are the lengths of the parallel sides. The bottom side has length 10 while the top (in terms of the angle θ) is $10+2\cdot 10\cos\theta$. The height h in terms of the angle θ is $10\sin\theta$. Hence the function we want to maximize is

$$A(\theta) = 100\sin\theta(1+\cos\theta), \theta \in [0, \pi/2].$$

Differentiating, we obtain

$$A' = 100(\cos\theta - \sin^2\theta + \cos^2\theta) = 100(2\cos^2\theta) + \cos\theta - 1)$$

using $\sin^2\theta + \cos^2\theta = 1$. This is a quadratic in $\cos\theta$ and solving for $\cos\theta$ gives

$$\cos\theta = \frac{-1 \pm \sqrt{1 - 4\cdot 2(-1)}}{2\cdot 2} = 1/2, -1$$

but -1 corresponds to θ outside $[0, \pi/2]$. Hence $\theta = \pi/3$ is our only critical number. This gives the maximum since

$$A(0) = 0,\ A(\pi/2) = 100,\ A(\pi/3) = 75\sqrt{3} = 129.9\ldots.$$

(Solution 2) Instead of parameterizing the area in terms of θ, we could use the height h. This gives

$$A(h) = \frac{1}{2}(10+10+2\sqrt{10^2-h^2})h = h(10+\sqrt{100-h^2}), h \in [0, 10].$$

Differentiating, we obtain

$$A' = 10 + \sqrt{100-h^2} - \frac{h^2}{\sqrt{100-h^2}} = 10 - \sqrt{100-h^2}\left(\frac{h^2}{100-h^2} - 1\right).$$

This doesn't exist at $h = 10$. Setting this equal to zero gives

$$\begin{aligned} 100 &= (100-h^2)\left(\frac{h^2}{100-h^2} - 1\right)^2, \\ &\ldots, \\ h &= \sqrt{75} = 5\sqrt{3}. \end{aligned}$$

Because $h = 5\sqrt{3} = 10\sin\theta$, this corresponds to an angle of $\theta = \pi/3$ (as above). This gives the max since

$$A(0) = 0, A(10) = 100, A(\sqrt{75}) = 129.9\ldots$$

as above.

3. (Bonus) Find the minimum length of the line segment from the y-axis to the x-axis going through the point (a, b) in the first quadrant $(a, b > 0)$. [Answer: $(a^{2/3} + b^{2/3})^{3/2}$.]

   (Solution 1) Let θ be the angle the line segment makes with the x-axis. Then the length l of the segment is the sum of two lengths, l_1 and l_2 (breaking the segement at the point (a, b)), where

   $$l_1 \cos\theta = a, \;\; l_2 \sin\theta = b$$

   so that the total length is

   $$l(\theta) = l_1 + l_2 = \frac{b}{\sin\theta} + \frac{a}{\cos\theta} = b\csc\theta + a\sec\theta, \;\; \theta \in (0, \pi/2).$$

   Differentiating we get

   $$\frac{dl}{d\theta} = -b\csc\theta\cot\theta + a\sec\theta\tan\theta$$

   and setting this equal to zero gives

   $$\tan\theta = \sqrt[3]{b/a}.$$

   Because $\lim_{\theta\to 0^+, \pi/2^-} l(\theta) = +\infty$ the minimum must occur here. To get the minimum value of l, note that the minimal line segment meets the x-axis at $x = a + b/\tan\theta$ and meets the y-axis at $y = b + a\tan\theta$ (draw a picture). Hence

   $$\begin{aligned} l^2 &= (a + b/\tan\theta)^2 + (b + a\tan\theta)^2 = (a + b/\sqrt[3]{b/a})^2 + (b + a\sqrt[3]{b/a})^2 \\ &= (a^{2/3} + b^{2/3})^3, \\ l &= (a^{2/3} + b^{2/3})^{3/2} \end{aligned}$$

   This is probably the easiest solution of the three.

(Solution 2) We parameterize the line by its slope, $m \in (-\infty, 0)$. The equation of the line with slope m through (a, b) is $y - b = m(x - a)$. When $x = 0$, $y = b - ma$ and when $y = 0$, $x = a - b/m$ (i.e. these are the legs of the right triangle whose hypotheneuse has length l we want to minimize). Hence the length of the line segment satisfies

$$l^2 = (b/m - a)^2 + (ma - b)^2 = (am - b)^2(1 + 1/m^2).$$

Taking derivatives gives

$$2l\frac{dl}{dm} = \frac{-2b}{m^2}\left(\frac{b}{m} - a\right) + 2a(ma - b) = (ma - b)(2a + \frac{2b}{m^3}).$$

Setting this equal to zero gives

$$m = b/a, -(a/b)^{1/3}.$$

Since the slope must be negative, we ignore the first solution. The length is then determined by

$$l^2 = (b^{2/3} + a^{2/3})^3, \ l = (a^{2/3} + b^{2/3})^{3/2}.$$

(Solution 3) Using similar triangles, we have $b/(x - a) = y/x$ where x and y are the legs of the right triangle whose hypotheneuse is length l we want to maximize. Hence $y = bx/(x - a)$ and

$$l^2 = x^2 + y^2 = x^2 + \left(\frac{bx}{x - a}\right)^2 = x^2\left(1 + \frac{b^2}{(x - a)^2}\right).$$

Differentiating gives

$$2l\frac{dl}{dx} = x^2\left(\frac{-2b^2}{(x - a)^3}\right) + \left(1 + \frac{b^2}{(x - a)^2}\right)2x = 2x\left(1 - \frac{ab^2}{(x - a)^3}\right).$$

Setting this equal to zero gives

$$x = a^{1/3}b^{2/3} + a,$$

and the minimal value of l can be found by evaluating at this value of x

$$l^2 = (a + a^{1/3}b^{2/3})^2\left(1 + \frac{b^2}{a^{2/3}b^{4/3}}\right) = (a^{2/3} + b^{2/3})^3, \ l = (a^{2/3} + b^{2/3})^{3/2}.$$

Don't worry, the problems on the exam will be much, much easier ☺