Math 1300-018 Quiz 7 Name:________________________

DUE MONDAY AT THE BEGINNING OF CLASS - SHOW ALL YOUR WORK. Use additional paper as necessary.

1. Differentiate the following functions:

   (a) $a^x + x^a, a > 0$ constant

   $$a^x \ln a + ax^{a-1}$$

   (b) $f(x)g(h(x))$

   $$f(x)g'(h(x))h'(x) + f'(x)g(h(x))$$

   (c) $\sqrt{x + \sqrt{x + \sqrt{x}}}$

   $$\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left(1 + \frac{1}{2\sqrt{x+\sqrt{x}}}\left(1 + \frac{1}{2\sqrt{x}}\right)\right)$$

   (d) $\dfrac{\log_2 x}{3^{x^2+x+1}}$

   $$\frac{1}{x \ln 2} 3^{x^2+x+1} + 3^{x^2+x+1} \ln 3(2x+1) \log_2 x$$

   (e) $f(x)^{g(x)}$

   $$f(x)^{g(x)} = e^{g(x)\ln f(x)}, \ \frac{d}{dx} e^{g(x)\ln f(x)} = f(x)^{g(x)}\left(g(x)\frac{f'(x)}{f(x)} + g'(x)\ln f(x)\right)$$

   (f) $x \ln x - x$

   $$\ln x$$

   (g) $\dfrac{(3x+2)^4(2x-3)^{1/4}}{(x+x^{-1})\ln x}$

   If y is the function above, take logarithms and differentiate:

   $$\frac{y'}{y} = \frac{12}{3x+2} + \frac{1}{2(2x-3)} - \frac{1}{x+x^{-1}}\left(1 - \frac{1}{x^2}\right) - \frac{1}{x \ln x}.$$

   Hence y' is y times the above.

   (h) $xe^x \tan x$

   $$x(e^x \tan x + e^x \sec^2 x) + e^x \tan x$$

   (i) $\dfrac{f(x)}{g(x) - h(x)}$

   $$\frac{(g(x) - h(x))f'(x) - f(x)(g'(x) - h'(x))}{(g(x) - h(x))^2}$$

(j) $7^{\sin(\pi z)}$

$$7^{\sin(\pi z)}(\ln 7)\pi\cos(\pi z)$$

(k) $\ln(\ln(\ln x))$

$$\frac{1}{\ln(\ln x)}\frac{1}{\ln x}\frac{1}{x}$$

(l) $\arccos(e^t) - \cos(e^{-t})$

$$\frac{-e^t}{\sqrt{1-e^{2t}}} - e^{-t}\sin(e^{-t})$$

(m) $\sqrt{y}^{\sqrt{y}}$
Let $z = \sqrt{y}^{\sqrt{y}}$. Then

$$\ln z = \frac{1}{2}\sqrt{y}\ln y,\quad \frac{z'}{z} = \frac{\sqrt{y}}{2y} + \frac{\ln y}{4\sqrt{y}},\quad z' = \sqrt{y}^{\sqrt{y}}\left(\frac{2+\ln y}{4\sqrt{y}}\right).$$

(n) $\left(\arctan\left(\frac{x}{2}\right)\right)^{2/3}$

$$\frac{2}{3}\left(\arctan\left(\frac{x}{2}\right)\right)^{-1/3}\cdot\frac{1}{1+\frac{x^2}{4}}\cdot\frac{1}{2}$$

(o) $\cos(\sin x)\tan(\cot x)$

$$(-\sin(\sin x)\cos x)\tan(\cot x) + \cos(\sin x)\sec^2(\cot x)(-\csc^2 x)$$

2. Consider the function

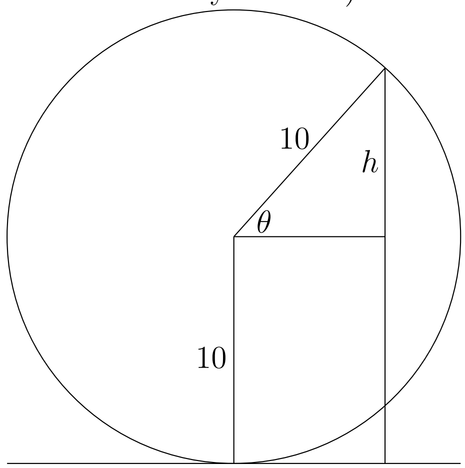
$$h(x) = (1+x)^p.$$

(a) Find the linearization (best linear approximation) of $h(x)$ near $x = 0$.

$$y = 1 + px$$

(b) For what values of p is this an overestimate or underestimate? [Hint: Consider the second derivative of h.]
The second derivative of h, $h'' = p(p-1)(1+x^{p-1})$ is positive at zero for $p > 1$ and $p < 0$, so that h is concave up and the tangent line lies below the graph of h. The second derivative is negative for $0 < p < 1$ so that h is concave down and the tangent line lines below the graph of h. For $p = 0, 1$ h is a line and the tangent line is exactly h itself.

3. A ferris wheel with a radius of 10 meters is rotating at a rate of one revolution every two minutes. How fast is a rider rising when his seat is 16 meters above the ground? (Assume the bottom of the ferris wheel is at ground level. Use the picture below or draw your own.)



With h and θ as in the picture, we know that

$$\frac{d\theta}{dt} = \frac{2\pi}{2} = \pi \ \frac{\text{radians}}{\text{minute}}$$

(because the wheel goes around once=2π every two minutes) and we want to find

$$\left.\frac{dh}{dt}\right|_{h=6 \text{ meters}}$$

how fast the height is increasing when the rider is 16 meters off the ground (i.e. when $h = 16 - 10 = 6$).

The quantities θ and h are related by

$$10 \sin\theta = h.$$

Differentiating with respect to time, using the chain rule because both h and θ are functions of time, we get

$$10 \cos\theta \frac{d\theta}{dt} = \frac{dh}{dt}.$$

When the rider is 16 meters above the ground, $h = 6$ and

$$\left.\cos\theta\right|_{h=6} = \frac{8}{10}$$

(we have a $(6, 8, 10)$ right triangle). Plugging in this value of $\cos\theta$ and the known value of $dh/dt = \pi$ we get

$$\left.\frac{dh}{dt}\right|_{h=6 \text{ meters}} = 10\frac{8}{10}\pi = 8\pi \ \frac{\text{meters}}{\text{minute}}$$

4. An $x \times y \times z$ box is growing. If

$$x(0) = 2,\ y(0) = 4,\ z(0) = 6 \text{ and } \left.\frac{dx}{dt}\right|_{t=0} = 3,\ \left.\frac{dy}{dt}\right|_{t=0} = 5,\ \left.\frac{dz}{dt}\right|_{t=0} = 7,$$

what is $\left.\dfrac{dS}{dt}\right|_{t=0}$, where S is the surface area of the box (all six sides)?
The surface area of the box is given by

$$S = 2(xy + xz + yz).$$

Differentiating with respect to time gives

$$\frac{dS}{dt} = 2\left(\frac{dx}{dt}y + x\frac{dy}{dt} + \frac{dx}{dt}z + x\frac{dz}{dt} + \frac{dy}{dt}z + y\frac{dz}{dt}\right).$$

At time zero, this gives $\frac{dS}{dt}|_{t=0} = 224$ square units per unit time.