Math 1300-018 Quiz 5 Name:________________________

1. Find the derivatives of the following functions. You need not simplify the result.

   (a) $2^x + x^2$

   $$2^x \ln 2 + 2x$$

   (b) $\sin(\sin(\sin x)))$
   This is a composition of three functions $f(g(h(x)))$. Using the chain rule, its derivative is a product of the derivatives of those three functions,

   $$f'(g(h(x)))g'(h(x))h'(x) = \cos(\sin(\sin x))\cos(\sin x)\cos x$$

   (c) $\dfrac{x^3}{\frac{x}{3} + \frac{3}{x}}$
   Directly applying the quotient rule, we have

   $$\frac{\left(\frac{x}{3} + \frac{3}{x}\right) 3x^2 - x^3 \left(\frac{1}{3} - \frac{3}{x^2}\right)}{\left(\frac{x}{3} + \frac{3}{x}\right)^2}.$$

   Or, simplifying the function first and then differentiating using the quotient rule gives

   $$f(x) = \frac{x^3}{\frac{x}{3} + \frac{3}{x}} = \frac{3x^4}{x^2 + 9}, \quad f'(x) = \frac{(x^2 + 9)12x^3 - 3x^4(2x)}{(x^2 + 9)^2}.$$

   (d) $x^5 e^{\tan x}$ Using the product rule and the chain rule, we have

   $$x^5 (e^{\tan x})' + e^{\tan x}(x^5)' = x^5 e^{\tan x} \sec^2 x + 5x^4 e^{\tan x}.$$

   (e) $\cos(x^2)\cos^2 x$
   Using the product rule and the chain rule, we have

   $$\cos(x^2)(\cos^2 x)' + \cos^2 x(\cos(x^2))' = \cos(x^2)(2\cos x \sin x) + \cos^2 x(2x\sin(x^2)).$$

2. For what values of r is $y = e^{rx}$ a solution of

   $$y'' - 4y' + y = 0?$$

   We have

   $$y = e^{rx}, \ y' = re^{rx}, \ y'' = r^2 e^{rx}$$

   so that the differential equation above becomes

   $$y'' - 4y' + y = r^2 e^{rx} - 4re^{rx} + e^{rx} = e^{rx}(r^2 - 4r + 1) = 0.$$

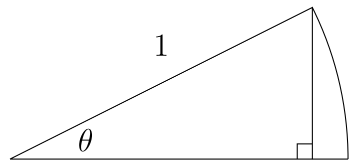
   Note that $e^{rx} > 0$ so that the above reduces to

   $$r^2 - 4r + 1 = 0.$$

   This has the two solutions (using the quadratic formula)

   $$r = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}.$$

3. Consider the diagram:



If $S(\theta)$ is the area of the sector and $T(\theta)$ is the area of the triangle, what is

$$\lim_{\theta\to 0^+}\frac{S(\theta)}{T(\theta)}?$$

We have

$$S(\theta)=\frac{\theta}{2\pi}\pi(1)^2=\frac{\theta}{2}$$

because we are considering a fraction $\frac{\theta}{2\pi}$ of the area of a circle of radius 1. We have

$$T(\theta)=\frac{1}{2}\cos\theta\sin\theta$$

because the triangle has base $\cos\theta$ and height $\sin\theta$. Hence the ratio of the areas is

$$\frac{S(\theta)}{T(\theta)}=\frac{\theta/2}{(\cos\theta\sin\theta)/2}=\frac{\theta}{\sin\theta\cos\theta}.$$

In the limit as $\theta\to 0^+$, we have

$$\lim_{\theta\to 0^+}\frac{S(\theta)}{T(\theta)}=\lim_{\theta\to 0^+}\frac{\theta}{\sin\theta\cos\theta}=\lim_{\theta\to 0^+}\frac{\theta}{\sin\theta}\lim_{\theta\to 0^+}\frac{1}{\cos\theta}=1$$

Using the fact that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1,$$

which we proved in order to find the derivatives of the trigonomtric functions. It's important that θ is measure in radians here. Using $\sin_{deg}(\theta)$ to represent the sine of an angle measured in degrees, we have (2π radians being 360 degrees)

$$\sin_{deg}(\theta)=\sin\left(\theta\frac{2\pi}{360}\right).$$

This would give

$$\lim_{\theta\to 0}\frac{\sin_{deg}(\theta)}{\theta}=\lim_{\theta\to 0}\frac{\sin\left(\theta\frac{2\pi}{360}\right)}{\theta}=\frac{2\pi}{360}\lim_{\phi\to 0}\frac{\sin\phi}{\phi}=\frac{\pi}{180}.$$