Math 1300-018 Quiz 3 Name:________________________

This is a take-home quiz, due Monday, September 14th in class. Work on your own and SHOW YOUR WORK AND EXPLAIN YOUR REASONING. Use additional paper as needed.

1. Consider the rational function

$$r(x) = \frac{3x^2 - 6x - 24}{x^4 - x^3 - 6x^2}.$$

(a) Where is $r(x)$ discontinuous?

We have

$$r(x) = \frac{3(x-4)(x+2)}{x^2(x-3)(x+2)}$$

so that $r$ has a removable discontinuity at $x = -2$ and vertical asymptotes of $x = 3, 0$

(b) For each value $x = a$ from (a), find $\lim_{x\to a^+} r(x)$ and $\lim_{x\to a^-} r(x)$ (including limits of $\pm\infty$).

At $x = -2$ we have

$$\lim_{x\to -2} r(x) = \lim_{x\to -2} \frac{3(x-4)}{x^2(x-3)} = \frac{3(-2-4)}{(-2)^2(-2-3)} = \frac{9}{10}.$$

At $x = 0$ we have

$$\lim_{x\to 0} r(x) = \lim_{x\to 0} \left(\frac{1}{x^2}\right)\left(\frac{3(x-4)(x+2)}{(x-3)(x+2)}\right) = +\infty$$

since the second quotient goes to 4 as $x \to 0$ while $x^2 \to 0^+$ as $x \to 0$.

As $x \to 3^+$ we have

$$\lim_{x\to 3^+} r(x) = \lim_{x\to 3^+} \left(\frac{1}{x-3}\right)\left(\frac{3(x-4)(x+2)}{x^2(x+2)}\right) = -\infty$$

since the second quotient goes to $-1/3$ as $x \to 3$ and $x - 3 \to 0^+$ as $x \to 3^+$.

As $x \to 3^-$ we have

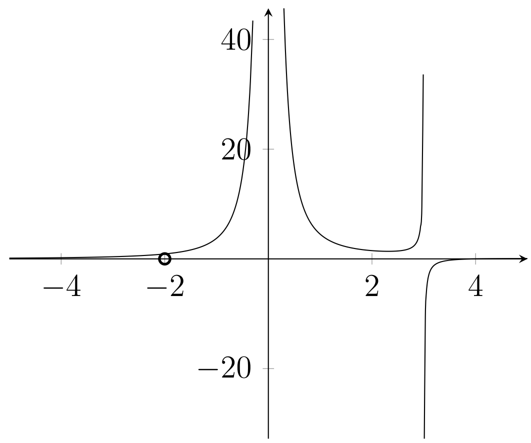
$$\lim_{x\to 3^-} r(x) = \lim_{x\to 3^-} \left(\frac{1}{x-3}\right)\left(\frac{3(x-4)(x+2)}{x^2(x+2)}\right) = +\infty$$

since the second quotient goes to $-1/3$ as $x \to 3$ and $x - 3 \to 0^-$ as $x \to 3^-$.

(c) Find $\lim_{x\to\infty} r(x)$.

$$\lim_{x\to\infty} \frac{3x^2 - 6x - 24}{x^4 - x^3 - 6x^2} = \lim_{x\to\infty} \frac{\frac{3}{x^2} - \frac{6}{x^3} - \frac{24}{x^4}}{1 - \frac{1}{x} - \frac{6}{x^2}} = \frac{0}{1} = 0.$$

(d) Sketch a graph of $r(x)$ including zeros and horizontal and vertical asymptotes.




2. Consider the following function (where $a, b \in \mathbb{R}$ are constants)

$$f(x) = \begin{cases} 3x - 2 & \text{if } x < -2 \\ ax^2 + bx + 1 & \text{if } -2 \leq x < 3 \\ ax + b & \text{if } x \geq 3 \end{cases}$$

(a) Determine $\lim_{x \to -2^-} f(x)$, $\lim_{x \to -2^+} f(x)$, $\lim_{x \to 3^-} f(x)$, and $\lim_{x \to 3^+} f(x)$ (your answers may include the constants $a$ and $b$).

We have

$$\begin{aligned} \lim_{x \to -2^-} f(x) &= \lim_{x \to -2^-} 3x - 2 = -8, \\ \lim_{x \to -2^+} f(x) &= \lim_{x \to -2^+} ax^2 + bx + 1 = a(-2)^2 + b(-2) + 1 = 4a - 2b + 1, \\ \lim_{x \to 3^-} f(x) &= \lim_{x \to 3^-} ax^2 + bx + 1 = a(3)^2 + b(3) + 1 = 9a + 3b + 1, \\ \lim_{x \to 3^+} f(x) &= \lim_{x \to 3^+} ax + b = a(3) + b = 3a + b. \end{aligned}$$

(b) Find values of $a$ and $b$ that will make $f$ continuous on $(-\infty, \infty)$. [Write down two equations in the two unknowns $a$ and $b$, one to describe continuity at $x = -2$ and another to describe continuity at $x = 3$.]

Note that $f$ is continuous everywhere except possibly at $x = -2, 3$ since it is defined by polynomials. At $x = -2, 3$ we must have the limit equal to the function value and $a, b$ must satisfy

$$-8 = 4a - 2b + 1, \ 9a + 3b + 1 = 3a + b$$

or (rewriting)

$$4a - 2b = -9, \ 6a + 2b = -1.$$

Adding the two together gives $10a = -10$ so that $a = -1$. Substituting this into either equation gives $b = 5/2$.

3. Use the intermediate value theorem to show that $x^2 = e^x$ has a solution (do not try to solve for $x$ exactly).

   At $x = 0$ we have $x^2 = 0$, $e^x = 1$ and $1 > 0$ (i.e. $f(x) = e^x - x^2 > 0$ at $x = 0$). At $x = -1$ we have $x^2 = 1$ and $e^x = 1/e$ with $1/e < 1$ (i.e. $f(x) = e^x - x^2 < 0$ at $x = -1$). Since the function $f(x)$ is continuous and takes on both positive and negative values (positive at 0, negative at $-1$ for instance), $f$ must take on the value 0 as well (in fact, somewhere on the interval $(-1, 0)$).

4. Find
$$\lim_{x\to\infty} \sqrt{x^2+ax} - \sqrt{x^2+bx}.$$

   [Hint: Get rid of the square roots using $x^2 - y^2 = (x-y)(x+y)$ then multiply and divide by an appropriate power of $x$. Your answer will be a finite number depending on the constants $a$ and $b$.]

   We have

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+ax} - \sqrt{x^2+bx} &= \lim_{x\to\infty} \left(\sqrt{x^2+ax} - \sqrt{x^2+bx}\right) \frac{\sqrt{x^2+ax} + \sqrt{x^2+bx}}{\sqrt{x^2+ax} + \sqrt{x^2+bx}} \\
&= \lim_{x\to\infty} \frac{(x^2+ax) - (x^2+bx)}{\sqrt{x^2+ax} + \sqrt{x^2+bx}} \\
&= \lim_{x\to\infty} \frac{(a-b)x}{\sqrt{x^2+ax} + \sqrt{x^2+bx}} \\
&= \lim_{x\to\infty} \frac{a-b}{\frac{1}{x}\sqrt{x^2+ax} + \sqrt{x^2+bx}} \\
&= \lim_{x\to\infty} \frac{a-b}{\sqrt{1+\frac{a}{x}} + \sqrt{1+\frac{b}{x}}} \\
&= \frac{a-b}{\sqrt{1}+\sqrt{1}} = \frac{a-b}{2}.
\end{aligned}$$