Math 1300-018 Quiz 12 Name:________________________

1. Find the area between the curves

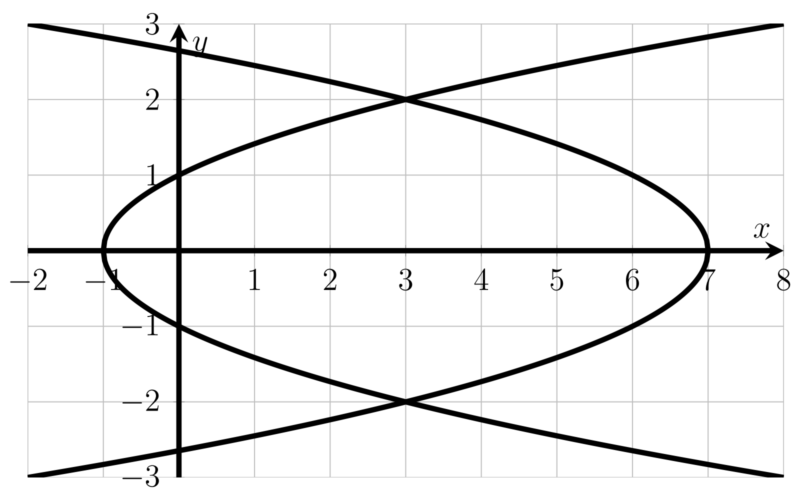
$$y^2 = x + 1,\ y^2 = 7 - x.$$

First you should draw a picture (see below). Viewing x as a function of y, we have two parabolas opening in opposite directions. Their points of intersection are

$$y^2 = x + 1 = 7 - x \Rightarrow x = 3,\ y = \pm 2.$$

We integrate with respect to y, between $y = -2, y = 2$, the difference between the x-values, $(7 - y^2) - (y^2 - 1)$:

$$\begin{aligned}\int_{-2}^{2} [(7 - y^2) - (y^2 - 1)]dy = \int_{-2}^{2} (-2y^2 + 8)dy &= \frac{-2}{3}y^3 + 8y\Big|_{-2}^{2} \\ &= (-16/3 + 16) - (16/3 - 16) = 64/3.\end{aligned}$$



2. Evaluate the following definite integrals

(a)

$$\int_{\pi/4}^{3\pi/4} \cot\theta d\theta = \int_{\pi/4}^{3\pi/4} \frac{\cos\theta}{\sin\theta} d\theta$$

Let $u = \sin\theta$, $du = \cos\theta d\theta$, so that $u(3\pi/4) = 1/\sqrt{2}$, $u(\pi/4) = 1/\sqrt{2}$. Then the above integral becomes

$$\int_{1/\sqrt{2}}^{1\sqrt{2}} \frac{du}{u} = 0 = \ln|u|\Big|_{1/\sqrt{2}}^{1/\sqrt{2}}.$$

[Or you could note that $\cot\theta$ has symmetry around $\pi/2$, $\cot(\pi/2+\theta) = -\cot(\pi/2-\theta)$.]

(b) $\int_1^e \frac{(\ln x)^3}{x} dx$

Let $u = \ln x$, $du = dx/x$ so that $u(1) = 0$, $u(e) = 1$. Then the above integral becomes

$$\int_0^1 u^3 du = \frac{u^4}{4}\Big|_0^1 = 1/4.$$

(c) $\int_0^{7/2} 3t(2t+1)^{1/3} dt$

Let $u = 2t+1$, $du = 2dt$, so that $t = \frac{u-1}{2}$, $u(0) = 1$, $u(7/2) = 8$. The above integral becomes

$$\begin{aligned}\int_1^8 \frac{3}{4}(u-1)u^{1/3} du &= \frac{3}{4}\int_1^8 (u^{4/3} - u^{1/3}) du = \frac{3}{4}\left(\frac{3u^{7/3}}{7} - \frac{3u^{4/3}}{4}\right)\Big|_1^8 \\ &= \frac{3}{4}\left(\left(\frac{3\cdot 2^7}{7}\right) - \frac{3\cdot 2^4}{4}) - \left(\frac{3}{7} - \frac{3}{4}\right)\right) = \frac{3627}{112}\end{aligned}$$

(d)

$$\int_{1/2}^{\sqrt{3}/2} \frac{dy}{\sqrt{1-y^2}} = \arcsin(y)\Big|_{1/2}^{\sqrt{3}/2} = \pi/3 - \pi/6 = \pi/6.$$

(e) $\int_0^{\ln 2} \frac{e^z dz}{e^{2z} + 2e^z + 1} = \int_0^{\ln 2} \frac{e^z dz}{(e^z+1)^2}$

Let $u = e^z + 1$, $du = e^z dz$ so that $u(0) = 2$, $u(\ln 2) = 3$. The above integral becomes

$$\int_2^3 \frac{du}{u^2} = -\frac{1}{u}\Big|_2^3 = \frac{1}{6}.$$

Or if you didn't see the factorization $e^{2z} + 2e^z + 1 = (e^z+1)^2$, let $u = e^z$, $du = e^z dz$ to get

$$\int_1^2 \frac{du}{u^2 + 2u + 1} = \int_1^2 \frac{du}{(u+1)^2} = -\frac{1}{u+1}\Big|_1^2 = \frac{1}{6}$$