Math 1081, Quiz 7 Name: ______________________________

Let $f(x, y) = x^4 + y^4 - 4xy + 1$.

1. Find the first order partial derivatives of f (f_x and f_y).

   $f_x = 4x^3 - 4y, f_y = 4y^3 - 4x$.

2. Find all critical points of f (solutions to the system of equations $f_x = f_y = 0$). There are three critical points.

   $f_x = 0 \Rightarrow y = x^3, f_y = 0 \Rightarrow x = y^3$. Substituting the second into the first gives $y = y^9$ which has solutions $y = 0, \pm 1$. Substituting these into $x = y^3$ gives the critical points $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

3. Find all second order partial derivatives of f (f_{xx},f_{yy}, and $f_{xy} = f_{yx}$).

   $f_{xx} = 12x^2, f_{yy} = 12y^2, f_{xy} = f_{yx} = -4$

4. Determine whether f has a local maximum, local minimum, or saddle point at each of the critical points from part (c) using the discriminant,

$$D_f(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2.$$

   We have $D_f(x, y) 144x^2y^2 - 16$. Evaluating this at the three critical points gives

$$D_f(0, 0) = -16 < 0, D_f(1, 1) = 128 > 0, D_f(-1, -1) = 128 > 0.$$

   So there is a saddle point at $(0, 0)$ and since $f_{xx}(1, 1) = f_{xx}(-1, -1) = 12 > 0$, f has a local minimum (of $f(1, 1) = f(-1, -1) = -1$) at both $(1, 1)$ and $(-1, -1)$.

   Below is the graph of f above the line $y = x$, with the minimums at $x = y = 1, x = y = -1$ and a saddle point at $x = y = 0$ (which looks like a maximum in this picture, but it is a minimum from the perpendicular direction).