Splitting factor maps into s- and u-bijective maps.

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1. History
2. Dynamical systems
3. Modelling
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Smale's program

Let M be a compact smooth manifold. Let $Diff(M)$ be the space of diffeomorphisms with the C^1 topology.

Question

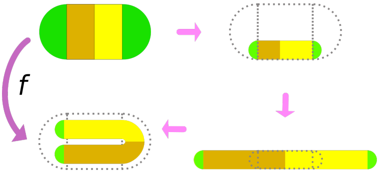
Can we find and classify, up to a reasonable equivalence relation, a subset that is a countable intersection of open dense sets in Diff(M)?

The classification should be manageable i.e there is a countable set of **invariants**.

Definition (Smale 1967)

Let M be a smooth manifold with a diffeomorphism $f : M \to M$, then (M, f) is an *Axiom A* system if the following two conditions hold:

1. The non-wandering set is hyperbolic and compact.
2. The periodic points are dense.

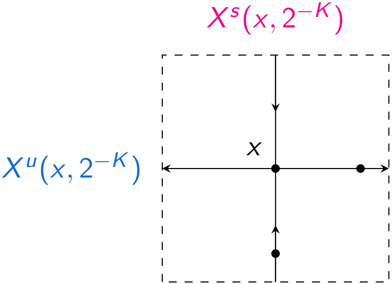


Ruelle defines a Smale space

A Smale space is a hyperbolic dynamical system (X, ϕ) where X is a compact metric space and ϕ is a homeomorphism.

$$\text{Hyperbolicity} \implies \text{local product structure}$$

i.e x in X given by local expanding and contracting directions.



Hyperbolic toral automorphism

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

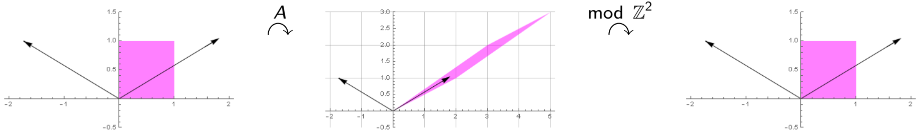
Define $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$f_A([x]) = [Ax]$$

where x is in $\mathbb{R}^2$ and $[x]$ denotes its equivalence class in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. By the integer components and the determinant, f_A is an invertible map.

Eigenvalues : $2+\sqrt{3}$ and $2-\sqrt{3}$.
A is *hyperbolic* $\sim$ none of its eigen-values lie on the unit circle.

Eigenvectors: $v_u = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ and $v_s = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$.



Notice $\mathbb{R}^2 = \{tv_u \mid t \in \mathbb{R}\} \oplus \{tv_s \mid t \in \mathbb{R}\} = E^u \oplus E^s$

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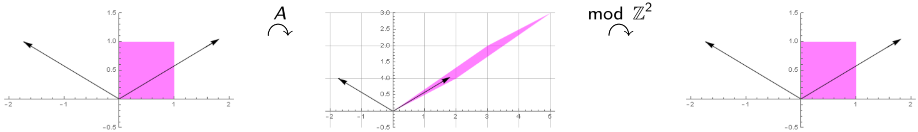

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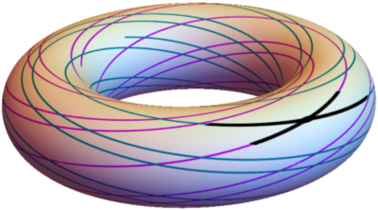
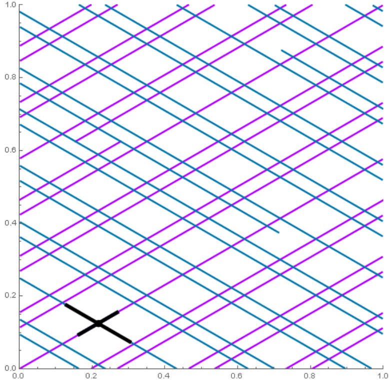
On a HTA the global unstable and stable sets wrap around densely.





The local stable and unstable sets are given by moving a little bit along the eigendirections. Locally, $\mathbb{T}^2$ can be viewed as $\mathbb{R} \times \mathbb{R}$.

Shifts of finite type

Let G be a finite directed graph which consists of a vertex set G^0, an edge set G^1, and two maps $r, s : G^1 \to G^0$. The source vertex of edge e is given by $s(e)$ and the range vertex is given by $r(e)$.

Definition

We define

$$\Sigma_G = \{(e_n)_{n\in\mathbb{Z}} \mid e_n \in G^1,\ r(e_n) = s(e_{n+1}) \text{ for all } n \text{ in } \mathbb{Z}\}$$

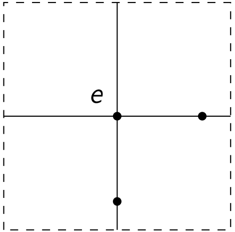
With the left shift map $\sigma : \Sigma_G \to \Sigma_G$,

$$\sigma(x)_n = e_{n+1}.$$

Let $e = (\ldots, e_{-1}, e_0, e_1, e_2, \ldots)$ and $\epsilon < 1/2$, then the local stable and unstable sets of e are given by,

$\Sigma_G^s(e, \epsilon)$

$\Sigma_G^u(e, \epsilon)$



$(\ldots, e_{-1}, e_0, *, *, \ldots)$

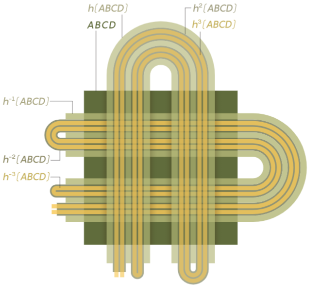
left tail equivalent

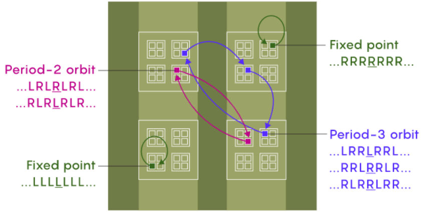
$(\ldots, *, *, e_1, e_2, \ldots)$

right tail equivalent

Smale's Horseshoe

$$(\Sigma_2 = \{0,1\}^{\mathbb{Z}}, \sigma) \Leftrightarrow (H, h)$$

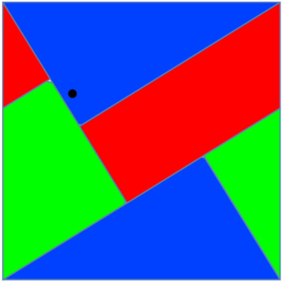




source: Quanta-How mathematicians make sense of chaos-David S. Richeson

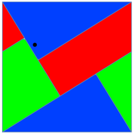
An HTA can be modeled using symbolic dynamics by way of Markov partitions, where $\pi : (\Sigma_G, \sigma) \to (\mathbb{T}^n, A)$ is a finite-to-one factor map.

Let x be in $\mathbb{T}^2$, how can we create a coding for this element?



Track the orbits of x.
Each rectangle corresponds to a vertex on the graph G.

$f_A^0(x)$



.B

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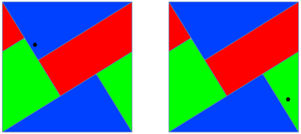

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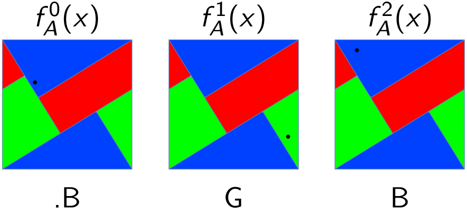

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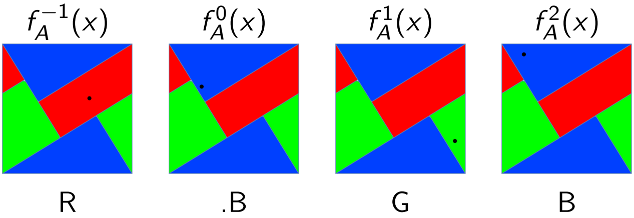

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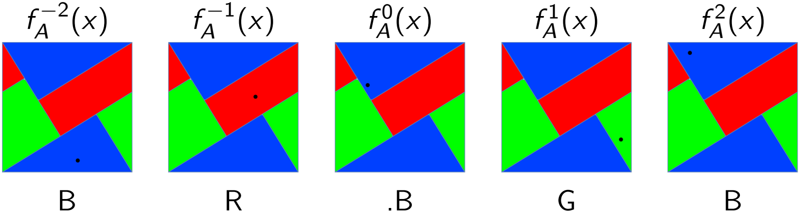
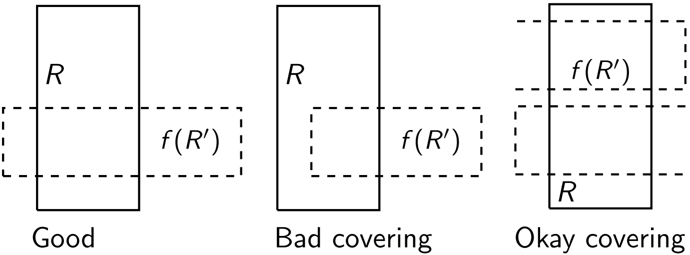


Markov Property



Theorem (Bowen 1970)

If (X, f) is an irreducible Smale space then (X, f) has Markov partitions. Equivalently, there is a shift of finite type, (Σ, σ) and finite-to-one surjective map $\pi : \Sigma \to X$ with the property that

$$\pi \circ f = \sigma \circ \pi$$

i.e a finite-to-one factor map.

$$\begin{array}{c} (\Sigma, \sigma) \\ \Big\downarrow \pi \\ (X, f) \end{array}$$

Fact

We say that $\pi : (X, f) \to (Y, g)$ *is an s-bijective map if for every* x *in* X*, the map* $\pi : X^s(x, \epsilon) \to Y^s(\pi(x), \epsilon')$ *is a local homeomorphism.*

A *u*-bijective map is defined and characterized analogously.

Given $(\mathbb{T}^d, f_A)$, we can find a factor map π.

$$\begin{array}{c} (\Sigma_G, \sigma) \\ \big\downarrow \pi \\ (\mathbb{T}^d, f_A) \end{array} \quad \text{locally represented as,} \quad \begin{array}{c} \text{Cantor} \times \text{Cantor} \\ \big\downarrow \pi \\ \mathbb{R}^m \times \mathbb{R}^n \cong E^s \times E^u \end{array}$$

where $m + n = d$.

Note: This map cannot be s-bijective nor u-bijective.

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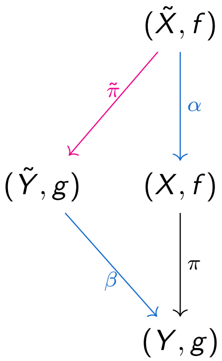
Theorem (Putnam 2005)

Let (X, d_x, f) and (Y, d_Y, g) be irreducible Smale spaces and suppose that

$$\pi : (X, f) \to (Y, g)$$

is an almost one-to-one factor map.
Then there exist irreducible Smale spaces, $(\tilde{X}, f)$ and $(\tilde{Y}, g)$ and factor maps $\alpha, \beta, \tilde{\pi}$ as shown.
Moreover, the diagram is commutative, α and β are u-resolving and $\tilde{\pi}$ is s-resolving.



Definition

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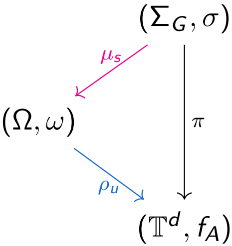
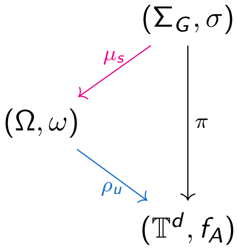
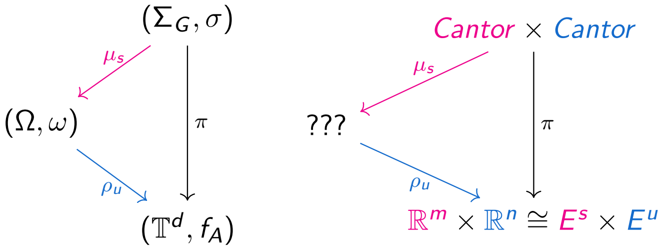


Suppose we have a splitting where μ_s, is an s-bijective map and ρ_u, a u-bijective map with a commutative diagram,



What must Ω look like locally? *Cantor* $\times \mathbb{R}^n$
What is a candidate space for Ω ?

Suppose we have a splitting where μ_s, is an *s*-bijective map and ρ_u, a *u*-bijective map with a commutative diagram,



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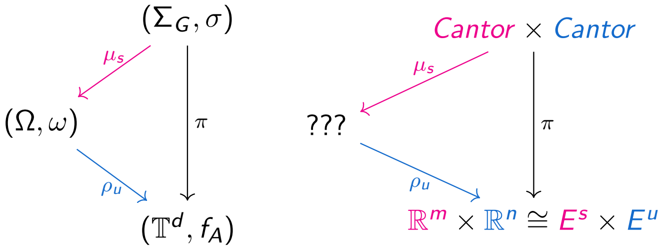

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What is a candidate space for Ω ?

Theorem (Williams 1968)

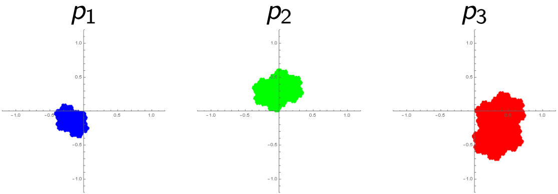
A solenoid is locally Cantor$\times\mathbb{R}^d$.

Theorem (Anderson and Putnam 1998)

A tiling space dynamical system which forces its border is topologically conjugate to a solenoid.

Substitution tiling systems, $(\Omega, \mathcal{P}, \omega)$

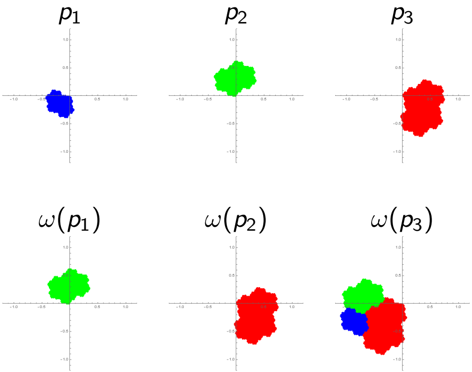
Prototiles, $\mathcal{P} = \{p_1, p_2 \ldots, p_n\}$. Each $p_i \subseteq \mathbb{R}^d$ is the closure of its interior.



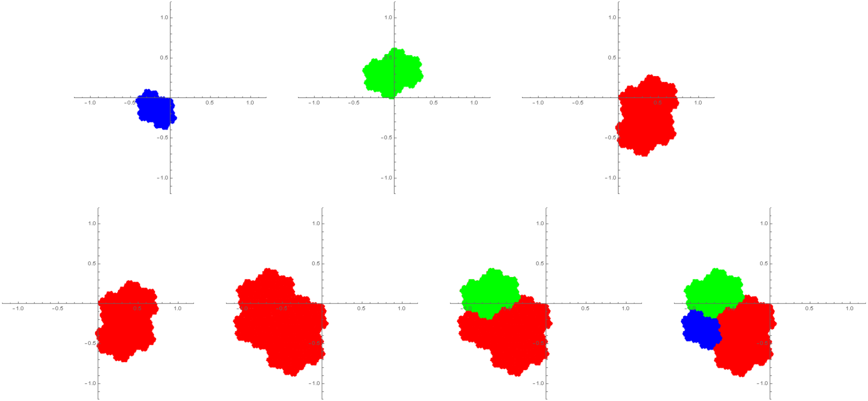
A tile t is a translation of some prototile.

A substitution rule $\omega(p_i)$ that inflates, possibly rotates, p_i, and subdivides with a finite set of translates of prototiles.

$$\omega(p_i) = \bigcup_{p_j \in \mathcal{P}} p_j + s_j$$

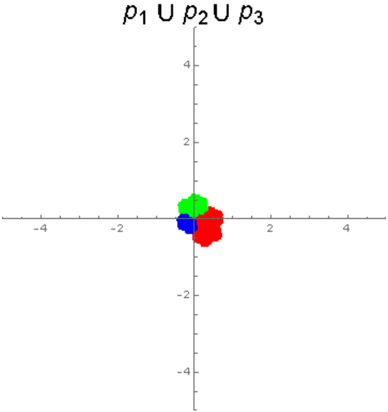


p_1 p_2 p_3



p_3 scale/rotate subdivide $\omega(p_3)$

A partial tiling is a collection of tiles whose interiors are pairwise disjoint.
A tiling is a partial tiling whose union is $\mathbb{R}^d$.



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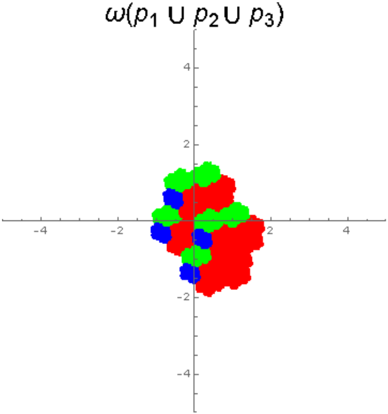

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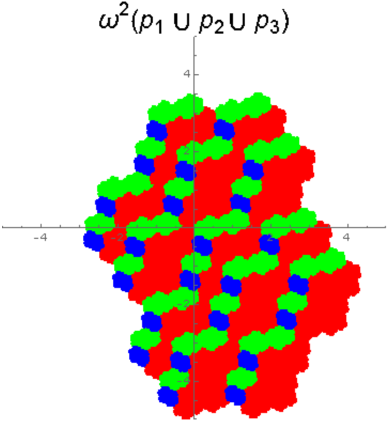
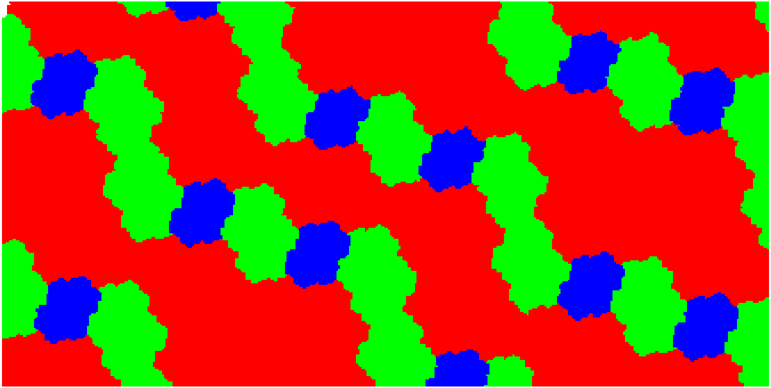


We define Ω to be the set of tilings that contain the patches of T.



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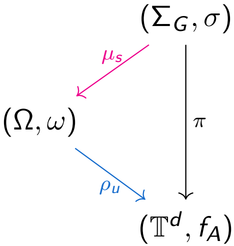


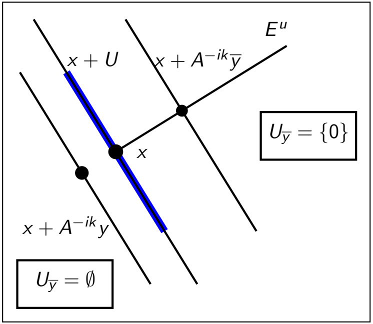
Main questions

1. Given a factor map from SFT to a Smale space, is there a condition for when/if the factor map splits?
2. What is the simplest SFT to use as a model? Can we find a factor map for such systems? Can we find a splitting for such systems?

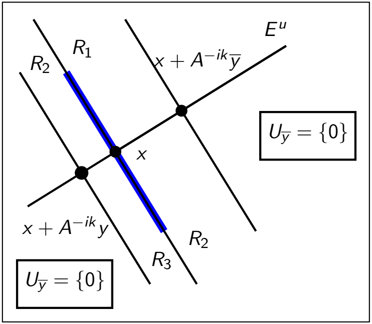
Focus: Consider an HTA as our Smale space.

Theorem (B, Putnam)

If a splitting for the factor map $\pi : \Sigma \to \mathbb{T}^d$ *exists then it must satisfy the Continuous Boundary Condition.*

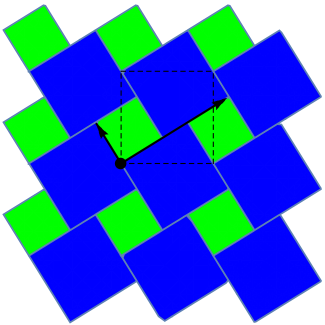


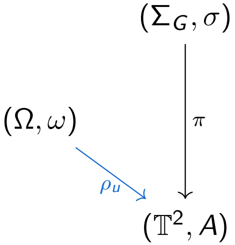
Condition fails.



Condition is satisfied.

Example for when a splitting does not exist, $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$





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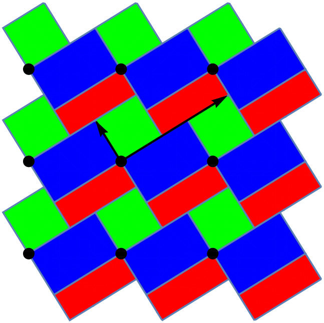

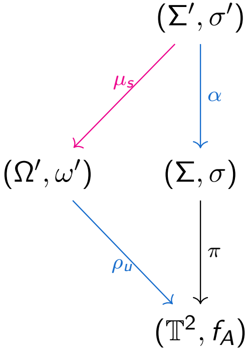

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Main questions

1. Given a factor map from SFT to a Smale space, is there a condition on if it splits? CBC fails $\rightarrow$ no splitting.
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   Can we find a splitting for such systems?

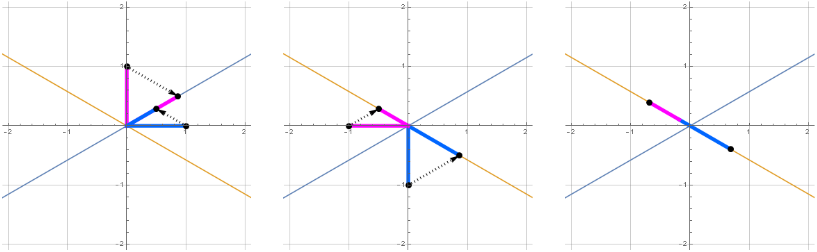
In the 2×2 case, for which HTAs does a splitting exist with the a factor map from a SFT defined by the same matrix?

Theorem (B, Putnam)

If A is hyperbolic, with $\det(A) = 1$, with positive entries and is not $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ then there exists a factor map from a SFT given by the matrix A^T, which has a splitting.

Constructing Markov partitions for $(\mathbb{T}^2, f_A)$

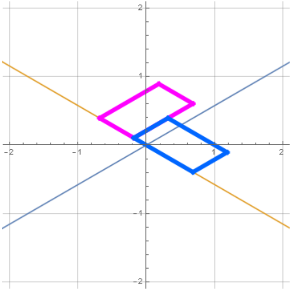
Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, then $v_u = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ and $v_s = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$.

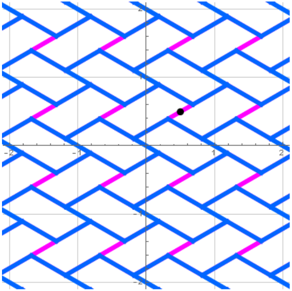


R_1^u, R_2^u R_1^s, R_2^s

Markov partition $\mathcal{M}$

$$\mathcal{M} = \{q(R_i^u + R_i^s) \mid 1 \leq i \leq 2\}$$



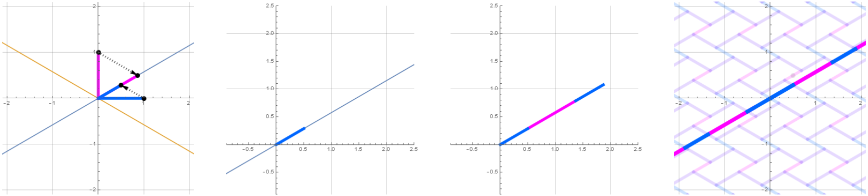


The tiling space (Ω, ω)

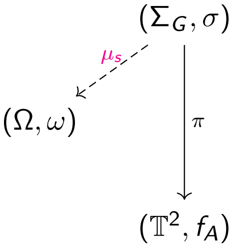
Prototiles: The projection of Markov partitions onto E^u
Substitution: Given by the Markov property.

$$\omega(R_1^u) = R_1^u \cup R_2^u + (1,0)^u \cup R_1^u + (1,1)^u$$

Protiles　　R_1^u　　$\omega(R_1^u)$　　tiling



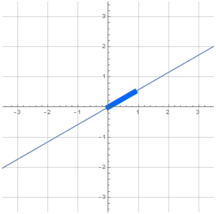
Tiling space to the SFT, $\rho_u : \Omega \to \mathbb{T}^2$



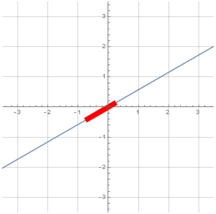
From the SFT to the tiling space, $\mu_s : \Sigma_G \to \Omega$

$$x = (\dots e_{-m}, \dots e_{-1}, \overbrace{e_0,}^{\textit{tile}} \overbrace{e_1, \dots e_n \dots}^{\textit{origin}})$$

$$T_0(x) = R^u_{r(e_0)} - \sum_{m=1}^{\infty} A^{-m} \nu(e_m)$$



$R^u_{r(e_0)}$

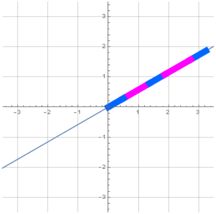


Initial tile $T_0(x)$

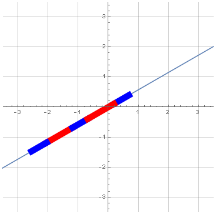
From the SFT to the tiling space, $\mu_s : \Sigma \to \Omega$

$$x = (\dots e_{-m}, \dots \overbrace{e_{-1}}^{patch}, \overbrace{e_0, e_1, \dots e_n \dots}^{shift})$$

$$T_1(x) = \omega(R^u_{r(x_{-1})}) - \sum_{m=0}^{\infty} A^{-m}\nu(x_m)$$



$\omega(R^u_{r(x_{-1})})$



Initial patch, $T_1(x)$

From the SFT to the tiling space, $\mu_s : \Sigma_G \to \Omega$

$$x = (\overbrace{\ldots e_{-m}, \ldots e_{-1}, e_0,}^{tiling} \overbrace{e_1, \ldots e_n \ldots}^{origin})$$

$$T_n(x) = \omega^n(R^u_{r(x_{-n})}) - \sum_{m=-n}^{\infty} A^{-m}\nu(x_m)$$

$$T_n(x) \subseteq T_{n+1}(x)$$

$$T(x) = \bigcup_{n=1}^{\infty} T_n(x)$$

The map $\mu_s : \Sigma_G \to \Omega$ only fails to be one-to-one when:

$$T_0(x) = T_0(x')$$

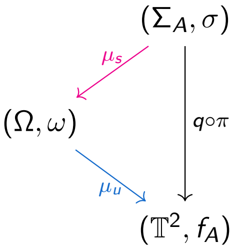
which only occurs when:

$$x = (\ldots, x_{n-1}, x_n, y, max, max, \ldots)$$
$$x' = (\ldots, x_{n-1}, x_n, y', min, min, \ldots)$$

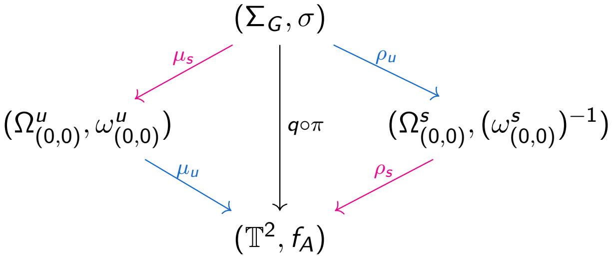
where y' is the successor of y in the ordering of the labels on the edges in E^u.

Theorem (B, Putnam)

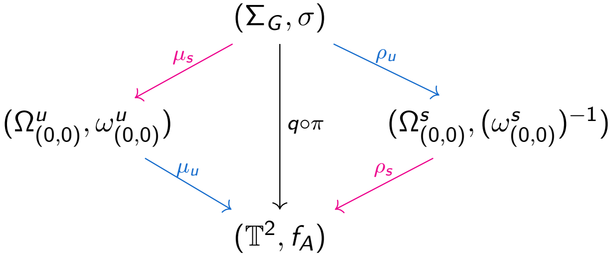
Let $\mu_s : \Omega \to \mathbb{T}^2$ be the Robinson map. Suppose T is in Ω, then $q \circ \pi(\mu_s^{-1}(T))$ contains a single point, which we call $\mu_u(T)$. Moreover, $\mu_u : \Omega \to \mathbb{T}^2$ is a continuous u-bijective factor map.



If A is not of the form $\begin{bmatrix} 1 & 1 \\ c & c+1 \end{bmatrix}$ for some c in $\mathbb{N}$, then π has a full splitting.

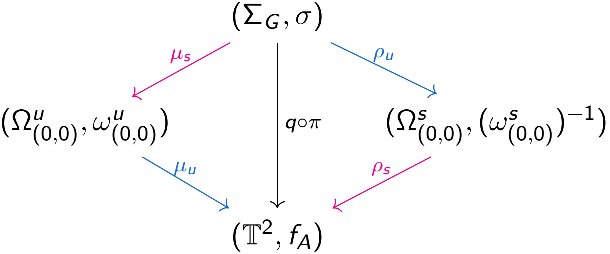


If A is not of the form $\left[\begin{smallmatrix} 1 & 1 \\ c & c+1 \end{smallmatrix}\right]$ for some c in $\mathbb{N}$ and A is not of the form $\left[\begin{smallmatrix} 1 & b \\ 1 & b+1 \end{smallmatrix}\right]$ for some b in $\mathbb{N}$ then π has a full splitting.



Only one matrix is of both forms, namely $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]$.

If A is not of the form $\begin{bmatrix} 1 & 1 \\ c & c+1 \end{bmatrix}$ for some c in $\mathbb{N}$ and A is not of the form $\begin{bmatrix} 1 & b \\ 1 & b+1 \end{bmatrix}$ for some b in $\mathbb{N}$ then π has a full splitting.



Only one matrix is of both forms, namely $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

Theorem (B, Putnam)

If A is hyperbolic, with $\det(A) = 1$, positive entries and is not $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ then there exists a factor map from a SFT given by the matrix A^T which has a splitting.

Many questions

1. What can we say about $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$? Note that $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^2$ has a splitting.
2. What about for dimension 3 or above?
3. We focused on HTAs, can this work be generalized to other Smale spaces?

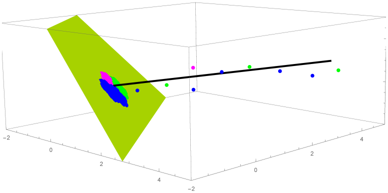
An example for dimension 3

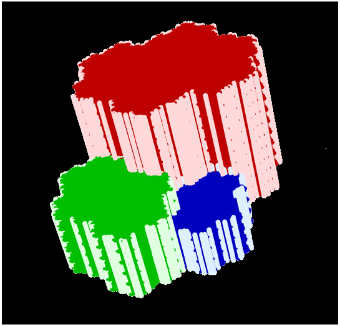
Let $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. The induced map f_B defines an HTA of $\mathbb{T}^3$.

Eigenvalues: $\beta > 1$, $\alpha, \overline{\alpha}$, where $\beta^3 - \beta^2 - \beta - 1 = 0$.

Expanding line and contracting plane.

The Markov partition is given by the following (viewed in $\mathbb{R}^3$).





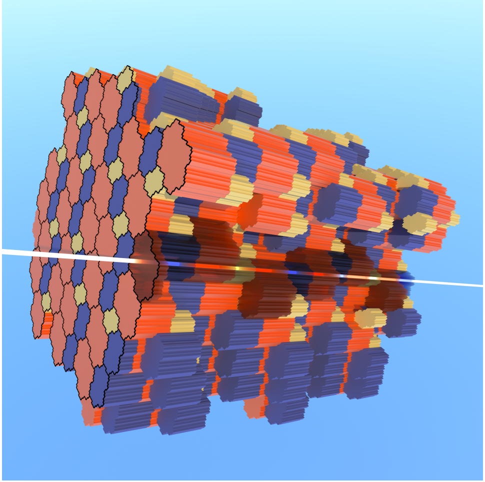
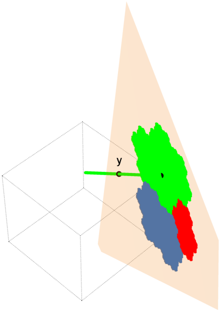
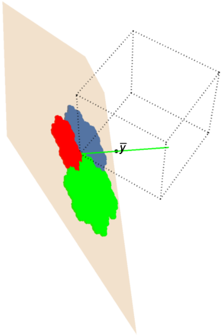


Image created by Edmund O. Harriss

Interior point



Boundary point



Condition fails: No splitting for the factor map exists.

"Of course the most rewarding part is the 'Aha' moment, the excitement of discovery and enjoyment of understanding something new- the feeling of being on top of a hill and having a clear view. But most of the time, doing mathematics for me is like being on a long hike with no trail and no end in sight." -Maryam Mirzakhani

Thank you for your attention!

Anderson, Jared E. and Ian F. Putnam (1998). "Topological invariants for substitution tilings and their associated C^*-algebras". In: *Ergodic Theory Dynam. Systems* 18.3, pp. 509–537. ISSN: 0143-3857. DOI: 10.1017/S0143385798100457. URL: http://dx.doi.org/10.1017/S0143385798100457.

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Putnam, Ian F. (2005). "Lifting factor maps to resolving maps". In: *Israel J. Math.* 146, pp. 253–280. ISSN: 0021-2172. DOI: 10.1007/BF02773536. URL: https://doi.org/10.1007/BF02773536.

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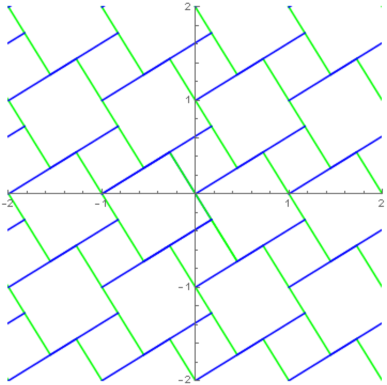
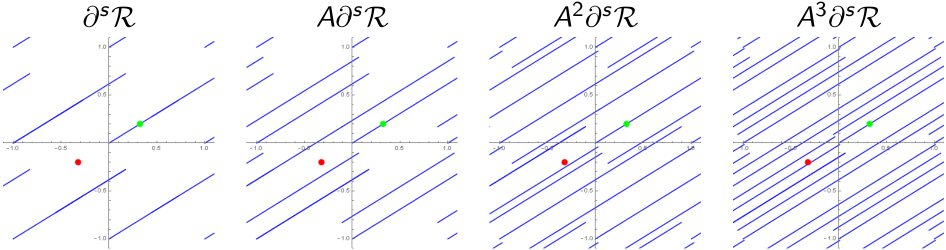
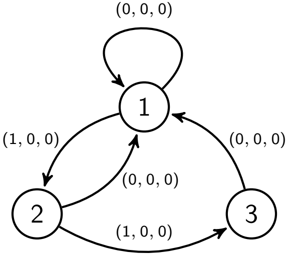
Williams, Robert F. (1968). "Classification of one dimensional attractors". In:

The boundary $\partial\mathcal{R} = \partial^s\mathcal{R} \cup \partial^u\mathcal{R}$



Our example does not satisfy Condition A.





Unstable (Tribonacci)

$$AR_1^u = R_1^u \cup R_2^u - (1,0,0)^s$$
$$AR_2^u = R_1^u \cup R_3^u - (1,0,0)^s$$
$$AR_3^u = R_1^u$$

Stable (Rauzy)

$$R_1^s = AR_1^s \cup AR_2^s \cup AR_3^s$$
$$R_2^s = AR_1^s + (1,0,0)^s$$
$$R_3^s = AR_2^s + (1,0,0)^s$$