Synchronizing Dynamical Systems

Andrew Stocker

CU Boulder

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Introduction

Goal: Study local properties of the bracket map (from the definition of a Smale space) and work with expansive dynamical systems where the bracket map is not necessarily defined everywhere.

Definition

A dynamical system (X, φ) is ***expansive*** if there exists a constant $\varepsilon_X > 0$ such that $d(\varphi^n(x), \varphi^n(y)) \leq \varepsilon_X$ for all $n \in \mathbb{Z}$ implies $x = y$.

C^*-Algebras From Expansive Dynamical Systems

- (Klaus Thomsen) $x, y \in X$ are called ***locally conjugate*** if there exist open neighborhoods U and V of x and y respectively, and a homeomorphism $\gamma : U \to V$ such that $\gamma(x) = y$ and

$$\lim_{n \to \pm\infty} \sup_{z \in U} d(\varphi^n(z), \varphi^n(\gamma(z))) = 0\,.$$

- For a Smale space these local conjugacies come from the bracket map. In other words, in a Smale space x and y are local conjugate if and only if they are homoclinic.

$$\gamma(z) = \left[\varphi^{-N}\left[\varphi^N[z, x], \varphi^N(y)\right], \varphi^N\left[\varphi^{-N}(y), \varphi^{-N}[x, z]\right]\right]$$

C^*-Algebras From Expansive Dynamical Systems

- This is an equivalence relation! Denote by $G^{\text{lc}}(X, \varphi) \subseteq X \times X$, however we topologize $G^{\text{lc}}(X, \varphi)$ with subbase

$$\{(z, \gamma(z)) \mid z \in U\}$$

for every local conjugacy $\gamma : U \to V$.

- With this topology $G^{\text{lc}}(X, \varphi)$ is an étale groupoid, we construct the groupoid C^*-algebra

$$A(X, \varphi) = C_r^*(G^{\text{lc}}(X, \varphi))$$

called the ***homoclinic algebra*** of (X, φ).

Heteroclinic Algebras

Remark: The ***heteroclinic algebras*** S and U are related C^*-algebras also constructed from an expansive dynamical system.

For Smale spaces these are exactly the *stable* and *unstable* algebras constructed by Putnam. (Note that Thomsen's construction of these algebras requires periodic points to be dense!)

Synchronizing Systems

- The local *stable* and *unstable* sets of $x \in X$ are defined as follows.

$$X^{\mathrm{s}}(x,\varepsilon) = \{y \in X \mid d\left(\varphi^n(x), \varphi^n(y)\right) \le \varepsilon \text{ for all } n \ge 0\}$$
$$X^{\mathrm{u}}(x,\varepsilon) = \left\{y \in X \mid d\left(\varphi^{-n}(x), \varphi^{-n}(y)\right) \le \varepsilon \text{ for all } n \ge 0\right\}$$

- For $0 < \varepsilon \le \frac{\varepsilon_X}{2}$ the intersection $X^{\mathrm{s}}(x,\varepsilon) \cap X^{\mathrm{u}}(y,\varepsilon)$ consists of at most one point (by expansiveness!). Define (Fried):

$$D_\varepsilon = \{(x,y) \in X \times X \mid X^{\mathrm{s}}(x,\varepsilon) \cap X^{\mathrm{u}}(y,\varepsilon) \ne \emptyset\}$$

and a map $[-,-] : D_\varepsilon \to X$ such that $[x,y] \in X^{\mathrm{s}}(x,\varepsilon) \cap X^{\mathrm{u}}(y,\varepsilon)$.

Notes:

- $[-,-]$ is continuous
- D_ε is closed and contains $\Delta_X = \{(x,x) \mid x \in X\}$.

Synchronizing Systems

Definition

A point $x \in X$ is called ***synchronizing*** if there exists $\delta_x > 0$ such that

$$X^{\mathsf{u}}(x, \delta_x) \times X^{\mathsf{s}}(x, \delta_x) \subseteq D_\varepsilon$$

and $[-, -]$ restricted to $X^{\mathsf{u}}(x, \delta_x) \times X^{\mathsf{s}}(x, \delta_x)$ is a homeomorphism onto its image, which is a neighborhood of x.

Definition

An expansive dynamical system (X, φ) is called a ***synchronizing system*** if it is *irreducible* and there exists a synchronizing point $x \in X$.

Remarks

- Smale spaces are synchronizing systems where every point is synchronizing.
- By irreducibility, synchronizing systems have a dense open set of synchronizing points.
- There exist expansive dynamical systems that are *not* synchronizing, e.g. *minimal* (every orbit is dense) expansive dynamical systems such as Toeplitz flows.
- *Synchronizing shifts* have been studied in symbolic dynamics.

C^*-Algebras From Synchronizing Systems

Theorem (Deeley, S.)

Let (X, φ) be a synchronizing system, then

1. the synchronizing points determine an ideal $\mathcal{I}_{\text{sync}} \subseteq A(X, \varphi)$, and
2. $A(X, \varphi)$ is asymptotically abelian.

Theorem (Deeley, S.)

If (X, φ) is a mixing finitely presented system, then

1. $S \otimes U$ is Morita equivalent to $\mathcal{I}_{\text{sync}}$,
2. $\mathcal{I}_{\text{sync}}$, S, and U are simple C^*-algebras.

Definition (D. Fried)

Finitely presented systems are synchronizing systems that can be covered by a finite number of "product neighborhoods".

Results: Dense Periodic Points

Theorem (Deeley, S.)

If (X, φ) is a synchronizing system then $\mathrm{Per}(X, \varphi)$ is dense in X.

Proof idea: Fix $x \in X$ where x is a synchronizing point, then from non-wandering can find y and $n > 0$ such that both y and $\varphi^n(y)$ are close enough to x to define $z_0 = [y, \varphi^n(y)]$. Then show the sequence

$$z_{k+1} = [\varphi^{-n}(z_k), \varphi^n(z_k)]$$

has a convergent subsequence. Use following theorem...

Adapted Metric

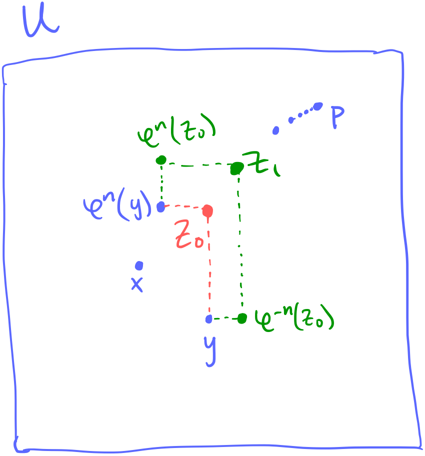
Theorem (D. Fried)

For any expansive dynamical (X, φ) system there exists a metric d and constants $\eta > 0$, $0 < \lambda < 1$ such that d is compatible with the topology on X and

$$d(\varphi(x), \varphi(y)) \leq \lambda d(x, y) \text{ for all } y \in X^{\mathrm{s}}(x, \eta) \text{ , and}$$
$$d(\varphi^{-1}(x), \varphi^{-1}(y)) \leq \lambda d(x, y) \text{ for all } y \in X^{\mathrm{u}}(x, \eta)\,.$$

Results: Dense Periodic Points

$$z_{k+1} = [\varphi^{-n}(z_k), \varphi^n(z_k)]$$



Shift Spaces

Let $\mathcal{A}$ be a finite set, consider the space $\mathcal{A}^{\mathbb{Z}} = \{(x_i)_{i \in \mathbb{Z}} \mid x_i \in \mathcal{A}\}$. The *shift map* $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by

$$\sigma(x)_i = x_{i+1}$$

Definitions

- A ***shift space*** is a closed subspace $X \subseteq \mathcal{A}^{\mathbb{Z}}$ which is invariant under σ.
- For a shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$, the set of finite words appearing in any element of X is denoted $\mathcal{L}(X) \subseteq \bigcup_{n \geq 0} \mathcal{A}^n$ and is called the *language* of X.

We think of (X, σ) as a dynamical system. Shift spaces are expansive!

Shift Spaces

Examples:

- Shift spaces that are also Smale spaces are called *shifts of finite type*. A shift of finite type can be constructed as the set of sequences in $\mathcal{A}^{\mathbb{Z}}$ which do not contain any of the words in a finite set of forbidden words denoted $\mathcal{F} \subseteq \bigcup_{n \geq 0} \mathcal{A}^n$.
  - The full 2-shift is $\{0, 1\}^{\mathbb{Z}}$.
  - The "golden mean shift" is the set of all sequences in $\{0, 1\}^{\mathbb{Z}}$ which do not contain consecutive 1's.
- Shift spaces that are also synchronizing are called *synchronizing shifts*. These have been studied in symbolic dynamics.
  - The even shift is the set of all sequences in $\{0, 1\}^{\mathbb{Z}}$ which have an even number of consecutive 0's between any 1's.

Topology on Shift Spaces

- In the topology on a shift space, two points x and y are close together if

$$x_{[-N,N]} = x_{-N}x_{-N+1}\cdots x_{N-1}x_N = y_{[-N,N]}$$

- Two points $x, y \in X$ are (un)stably equivalent in a shift space if for some $N \in \mathbb{Z}$

$$x_n = y_n \text{ for all } n \geq N \text{ (stable)}$$
$$x_n = y_n \text{ for all } n \leq N \text{ (unstable)}$$

- The local stable and unstable sets are

$$X^s(x,\varepsilon) = \{y \in X \mid y_n = x_n \text{ for all } n \geq N_\varepsilon\}$$
$$X^u(x,\varepsilon) = \{y \in X \mid y_n = x_n \text{ for all } n \leq N_\varepsilon\}$$

Shift Spaces

- Let $x \in X$ be synchronizing, $\delta_x > 0$ such that for $y \in X^u(x, \delta_x), z \in X^s(x, \delta_x)$ we have the following:

$$y = (\dots x_{-N-2}x_{-N-1})(x_{-N}x_{-N+1}\dots x_{N-1}x_N)(y_{N+1}y_{N+2}\dots)$$
$$z = (\dots z_{-N-2}z_{-N-1})(x_{-N}x_{-N+1}\dots x_{N-1}x_N)(x_{N+1}x_{N+2}\dots)$$
$$\downarrow$$
$$[y, z] = (\dots z_{-N-2}z_{-N-1})(x_{-N}x_{-N+1}\dots x_{N-1}x_N)(y_{N+1}y_{N+2}\dots)$$

- A synchronizing word in a shift space X is a word w such that if u, v are words such that $uw, wv \in \mathcal{L}(X)$, then $uwv \in \mathcal{L}(X)$. Synchronizing shifts are irreducible shift spaces which contain a synchronizing word.

Shift Spaces

Local conjugacy relation for shift spaces:

- Two points $x, y \in X$ are locally conjugate if for some N large enough, there is a bijection constructed as follows. Let z satisfy $z_{[-N,N]} = x_{[-N,N]}$.

$$z = (\dots z_{-N-2} z_{-N-1})(x_{-N} x_{-N+1} \dots x_{N-1} x_N)(z_{N+1} z_{N+2} \dots)$$
$$\downarrow \gamma$$
$$\gamma(z) = (\dots z_{-N-2} z_{-N-1})(y_{-N} y_{-N+1} \dots y_{N-1} y_N)(z_{N+1} z_{N+2} \dots)$$

- (Krieger) We can construct the homoclinic algebra of a shift space by considering equivalence classes of words arising from the bijection above.

Even Shift

Let $X \subseteq \{0,1\}^{\mathbb{Z}}$ be the set of all elements of $\{0,1\}^{\mathbb{Z}}$ which do not contain the word $10^{2k+1}1$ for any $k \geq 0$. This is a shift space called the ***even shift***.

The even shift is a *sofic shift* (not a Smale space!).

Consider the sequence of all zeros:

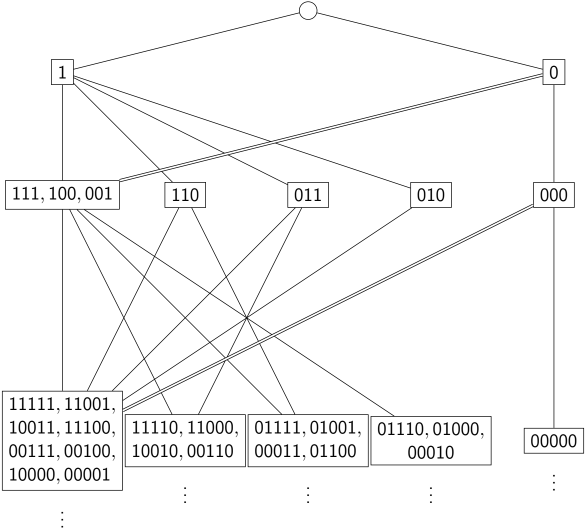
$$\overline{0} = \ldots 00000 \ldots \in X$$

This point is *not* synchronizing! Let $x \in X^{\mathrm{u}}(\overline{0}, \varepsilon)$ and $y \in X^{\mathrm{s}}(\overline{0}, \varepsilon)$.

$$x = \ldots 001000000 \ldots$$
$$y = \ldots 000000100 \ldots$$
$$[x, y] = \ldots 001000100 \ldots$$

Even Shift

Bratteli diagram for the homoclinic algebra of the even shift:



Even Shift

- The Bratteli diagram determines the homoclinic algebra $A(X,\sigma)$ of the even shift. In particular $A(X,\sigma)$ is an AF-algebra (for all shift spaces).
- The K-theory of A can be computed as

$$K_0(A) = \varinjlim \left\{ \mathbb{Z}^5 \xrightarrow{P} \mathbb{Z}^5 \xrightarrow{P} \dots \right\}$$

where

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is the matrix encoding the edge relations in the Bratteli diagram.

Even Shift

For the even shift, the stable algebra S and the unstable algebra U can be computed via Bratteli diagrams similar to the previous one, where

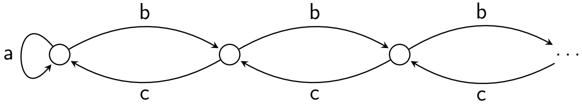
$$P^s = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad P^u = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The ideal $\mathcal{I}_{\text{sync}}$ in the even shift is built from the vertices in the Bratteli diagram representing equivalence classes of synchronizing words. In particular we have the following.

$$P_{\text{sync}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = P^s_{\text{sync}} \otimes P^u_{\text{sync}}$$

Infinite Rank Example

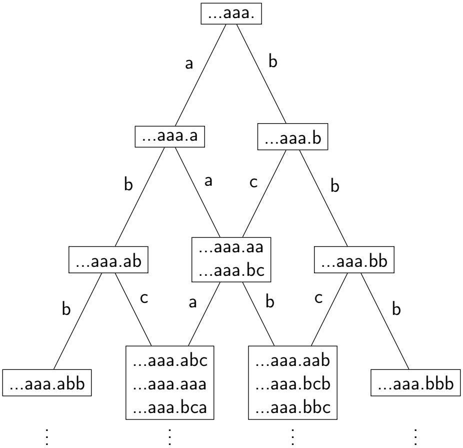
Let $X \subseteq \{a, b, c\}^{\mathbb{Z}}$ be the closure of the set of bi-infinite paths on the following graph



This is a synchronizing shift: for example the word 'a' is synchronizing.

Infinite Rank Example

Bratteli diagram* for stable algebra of X (with respect to the periodic point $\overline{a} = ...aaa.aa...$).



Results

If (X, φ) is a synchronizing system and $x, y \in X$ are locally conjugate, then x is synchronizing if and only if y is synchronizing. Hence we have an ideal $\mathcal{I}_{\text{sync}} \subseteq A$ and a short exact sequence

$$0 \longrightarrow \mathcal{I}_{\text{sync}} \longrightarrow A \longrightarrow A/\mathcal{I}_{\text{sync}} \longrightarrow 0$$

The ideal $\mathcal{I}_{\text{sync}}$ has similar properties to a Smale space C^*-algebra.

- For example the even shift has only one non-synchronizing point, and $A/\mathcal{I}_{\text{sync}} \cong \mathbb{C}$.
- Expansive homeomorphisms on surfaces have only a finite number of non-synchronizing points, and so $A/\mathcal{I}_{\text{sync}}$ is a finite dimensional C^*-algebra.

Results

We can think of the K-theory of the C^*-algebras A, S, and U as giving information about what type of expansive system (X, φ) is.

Theorem (S.)

For an expansive dynamical system (X, φ), if $S \otimes U$ is not Mortia equivalent to A then (X, φ) is not a Smale space.

- For example, this is not true for the even shift, so it cannot be a shift of finite type.

Theorem (S.)

For a shift space (X, σ), if the rank of $K_0(A)$ is not finite then X cannot be a sofic shift.

- The $a^n b^n$-shift has infinite rank K-theory, and is not a sofic shift.

References

Thank you!

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