An introduction to Smale spaces and their groupoids

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Smale space conference

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Plan

1. Intro and Example: A hyperbolic toral automorphism
2. Introduction to Smale spaces
3. Smale's decomposition theorem and Bowen's theorem
4. Groupoids associated with Smale spaces
5. C^*-algebras of Smale spaces
6. Wieler solenoids

Intro

A Smale space consists of a compact metric space and a homeomorphism (X, φ) with canonical expanding and contracting directions.

Smale spaces were introduced by David Ruelle and include:

- Shifts of finite type (Smale's horseshoe),
- Hyperbolic toral automorphisms,
- Solenoids (Bob Williams and Susie Weiler):
  - Dynamical systems associated with certain substitution tiling spaces (Anderson and Putnam),
  - Solenoids associated to the limit space of a contracting self-similar group (Nekrashevych),
- the basic sets of Smale's Axiom A systems (Ruelle).

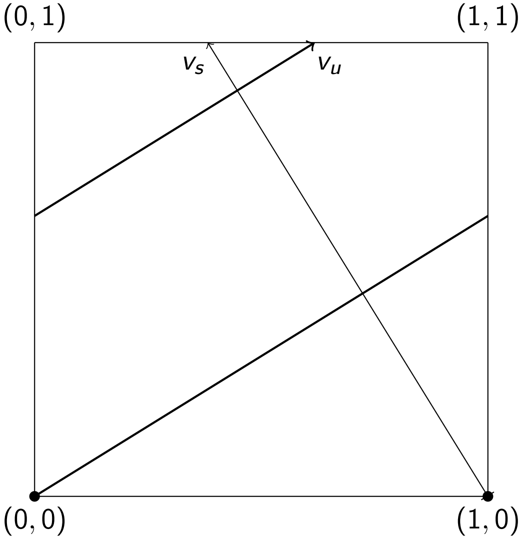
A Hyperbolic Toral Automorphism

Let A be the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

- $A : \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism, where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.
- Let $\gamma = \frac{1+\sqrt{5}}{2}$ be the golden mean.
- The eigenvalues for A are $\gamma > 1$ and $-\gamma^{-1}$ corresponding with eigenvectors

$$v_u = \begin{pmatrix} \gamma \\ 1 \end{pmatrix} \qquad \text{and} \qquad v_s = \begin{pmatrix} 1 \\ -\gamma \end{pmatrix}.$$



Hyperbolic Toral Automorphism

- Consider the dynamical system $(\mathbb{T}^2, A)$ and let x, y be in $\mathbb{T}^2$. We aim to define equivalence relations with respect to the homeomorphism A.
  - Stable equivalence:

$$x \sim_s y \iff \lim_{n\to\infty} d(A^n(x), A^n(y)) = 0.$$

  - Let $X^s(x)$ denote the stable equivalence class of x and observe:

$$X^s(x) = \{x + tv_s (\mathrm{mod}\mathbb{Z}^2) \mid t \in \mathbb{R}\}.$$

  - For $0 < \varepsilon < 1/2$, define the local stable equivalence class of x:

$$X^s(x, \varepsilon) = \{x + tv_s (\mathrm{mod}\mathbb{Z}^2) \mid |t| < \varepsilon\}.$$

- Similarly, unstable equivalence:

$$x \sim_u y \iff \lim_{n\to\infty} d(A^{-n}(x), A^{-n}(y)) = 0.$$

- Let $X^u(x)$ denote the unstable equivalence class of x and observe:

$$X^u(x) = \{x + tv_u (\mathrm{mod} \mathbb{Z}^2) \mid t \in \mathbb{R}\}.$$

- For $0 < \varepsilon < 1/2$, define the local unstable equivalence class of x:

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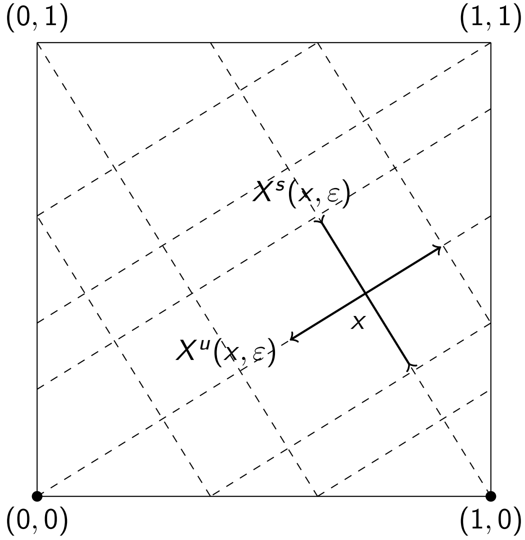


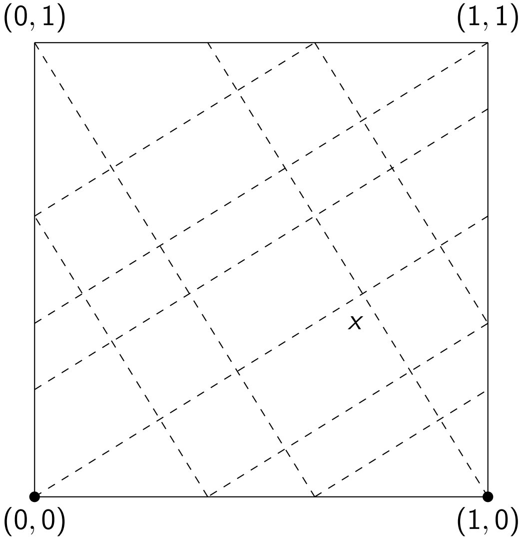
Local and global equivalence classes of x

- Homoclinic equivalence:

$$x \sim_h y \iff x \sim_s y \text{ and } x \sim_u y.$$

- Let $X^h(x)$ denote the homoclinic equivalence class of x and observe:

$$X^h(x) = X^s(x) \cap X^u(x).$$



The points where the dashed lines intersect are homoclinic to x

General definition of a Smale space

A Smale space is a compact metric space (X, d) with a homeomorphism $\varphi : X \to X$ such that there exist constants $\varepsilon_X > 0, \lambda > 1$ and bracket map

$$(x, y) \in X, d(x, y) \leq \varepsilon_X \mapsto [x, y] \in X$$

satisfying:

- B1 $[x, x] = x$,
- B2 $[x, [y, z]] = [x, z]$,
- B3 $[[x, y], z] = [x, z]$,
- B4 $\varphi[x, y] = [\varphi(x), \varphi(y)]$;

for any x, y, z in X, where both sides of the equality are defined.

For each x in X and $0 < \varepsilon \leq \varepsilon_X$, define sets

$$X^s(x, \varepsilon) = \{y \in X \mid d(x, y) \leq \varepsilon, [y, x] = x\},$$
$$X^u(x, \varepsilon) = \{y \in X \mid d(x, y) \leq \varepsilon, [x, y] = x\}.$$

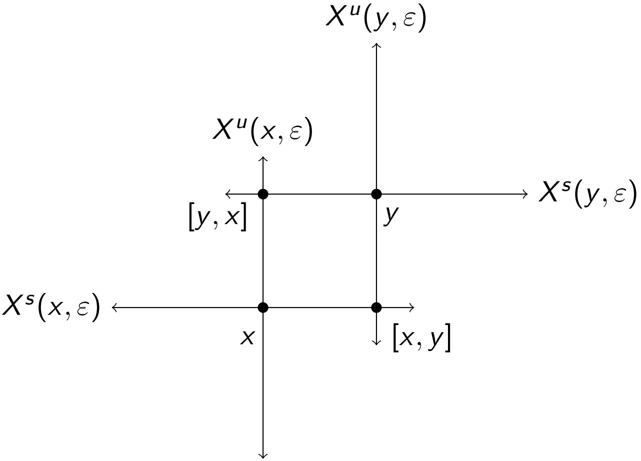
The two final axioms are

C1 For y, z in $X^s(x, \varepsilon_X)$, we have

$$d(\varphi(y), \varphi(z)) \leq \lambda^{-1} d(y, z),$$

C2 For y, z in $X^u(x, \varepsilon_X)$, we have

$$d(\varphi^{-1}(y), \varphi^{-1}(z)) \leq \lambda^{-1} d(y, z).$$



The bracket map

Note that if a bracket map exists on (X, φ) then it's unique.

We typically study Smale spaces with topological recurrence conditions. Let (X, φ) be a dynamical system, then:

- A point x in X is called *non-wandering* if for every open set U containing x, there is a positive integer N such that $\varphi^N(U) \cap U$ is non-empty. We say that (X, φ) is non-wandering if every point of X is non-wandering.

- (X, φ) is said to be *irreducible* if, for every (ordered) pair of non-empty sets U, V, there is a positive integer N such that $\varphi^N(U) \cap V$ is non-empty.

- (X, φ) is said to be *mixing* if, for every (ordered) pair of non-empty sets U, V, there is a positive integer N such that $\varphi^n(U) \cap V$ is non-empty for all $n \geq N$.

Smale's Decomposition Theorem

Theorem (Smale's Decomposition Theorem)

Let (X, φ) be a Smale space.

- *If (X, φ) is non-wandering, then there exists a partition of X into a finite number of clopen, pairwise disjoint subsets, each of which is invariant under φ and so that the restriction of φ to each is irreducible. Moreover, this decomposition is unique.*
- *If (X, φ) is irreducible, then then there exists a partition of X into a finite number of clopen, pairwise disjoint subsets which are cyclicly permuted by φ. If the number of these sets is N, then φ^N (which leaves each invariant) is mixing on each element of the partition.*

Markov partitions and Bowen's Theorem

Theorem (Bowen 1970)

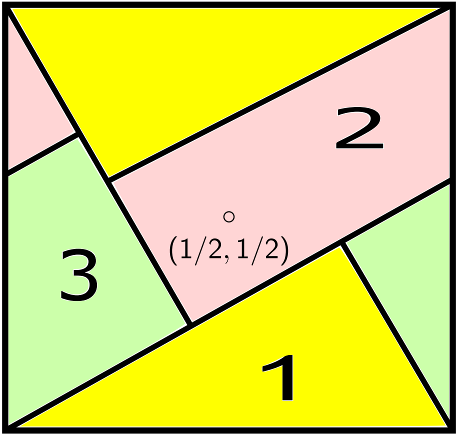
Suppose (X, φ) is an irreducible Smale space. Then there is a shift of finite type (Σ, σ) and a factor map $\pi : (\Sigma, \sigma) \to (X, \varphi)$ such that π is finite-to-one and one-to-one on a dense G_δ subset of Σ.

A non-empty subset $R \subset X$ is called a *rectangle* if $\mathrm{diam}(R) \leq \varepsilon_X$ and $[x, y] \in R$, for any $x, y \in R$.

Bowen's proof showed that irreducible Smale spaces admit Markov partitions of arbitrarily small diameter; that is, the space X can be partitioned into closed rectangles that respect the dynamical structure and overlap only on their boundaries.

Markov partitions and Bowen's Theorem

Encoding the point $(1/2, 1/2) \in \mathbb{T}^2$ as a sequence in (Σ_3, σ):



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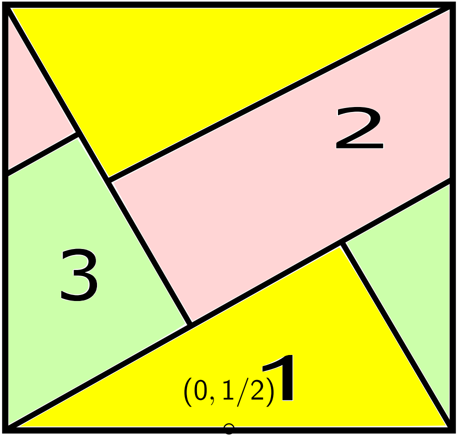

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Markov partitions and Bowen's Theorem

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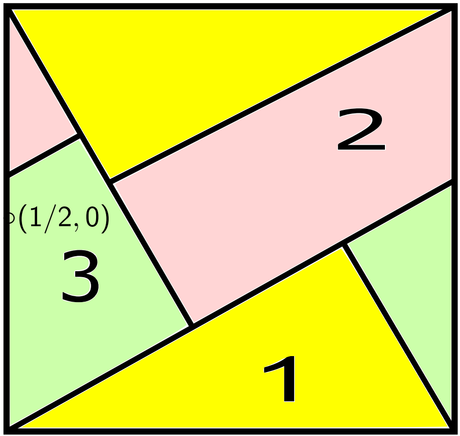

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Markov partitions and Bowen's Theorem

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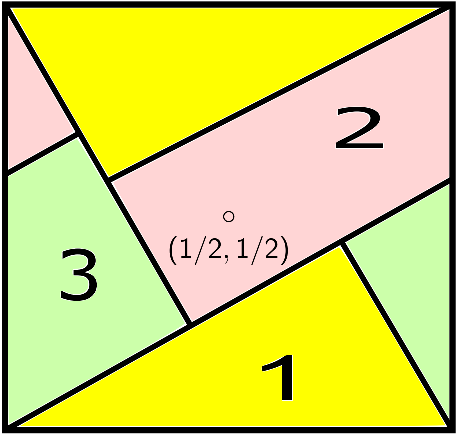

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Markov partitions and Bowen's Theorem

For a Smale space (X, φ), suppose $\mathcal{R}_1$ is a Markov partition with sufficiently small diameter.

The Markov property implies that we can find Markov partitions $\mathcal{R}_n$, for $n \geq 2$, where each $\mathcal{R}_n$ refines $\mathcal{R}_{n-1}$ and $\text{diam}(\mathcal{R}_n)$ goes to zero.

The sequence $(\mathcal{R}_n)_{n \in \mathbb{N}}$ is the main ingredient for deriving Bowen's factor map.

Groupoids (Ruelle, Putnam-Spielberg)

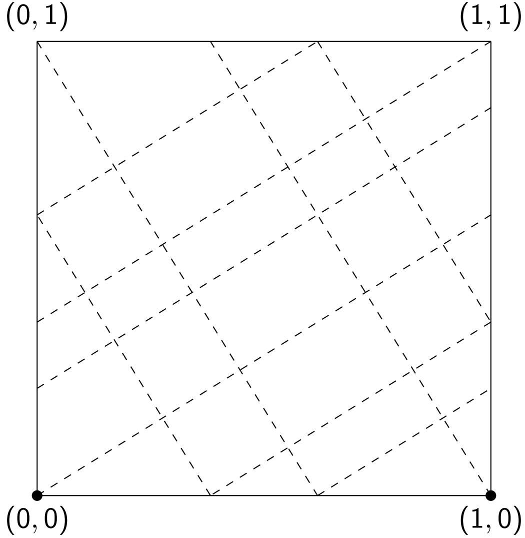
- Define global equivalence relations as follows:

$$X^s(x) = \{y \in X \mid d(\varphi^n(x), \varphi^n(y)) \to 0 \text{ as } n \to \infty\},$$
$$X^u(x) = \{y \in X \mid d(\varphi^{-n}(x), \varphi^{-n}(y)) \to 0 \text{ as } n \to \infty\}.$$

- Let (X, φ) be a Smale space and let P be a periodic orbit. Let

$$X^s(P) = \bigcup_{p \in P} X^s(p)$$
$$X^u(P) = \bigcup_{p \in P} X^u(p)$$
$$X^h(P) = X^s(P) \cap X^u(P).$$

- Under mild hypotheses, $X^h(P)$ is countable and dense.



Hyperbolic Toral Automorphism: $X^h(\{(0,0)\})$

- Stable and unstable equivalence leads to groupoids on (X, d, φ):

$$G^s(X, \varphi, P) = \{(v, w) | v \sim_s w \text{ and } v, w \in X^u(P)\}$$
$$G^u(X, \varphi, P) = \{(v, w) | v \sim_u w \text{ and } v, w \in X^s(P)\}$$
$$G^h(X, \varphi) = \{(v, w) | v \sim_u w \text{ and } v \sim_s w\}.$$

- These are called the stable, unstable and homoclinic groupoids of a Smale space.

- These groupoids are independent of P, up to groupoid equivalence (meaning the C^*-algebras will be strongly Morita equivalent).

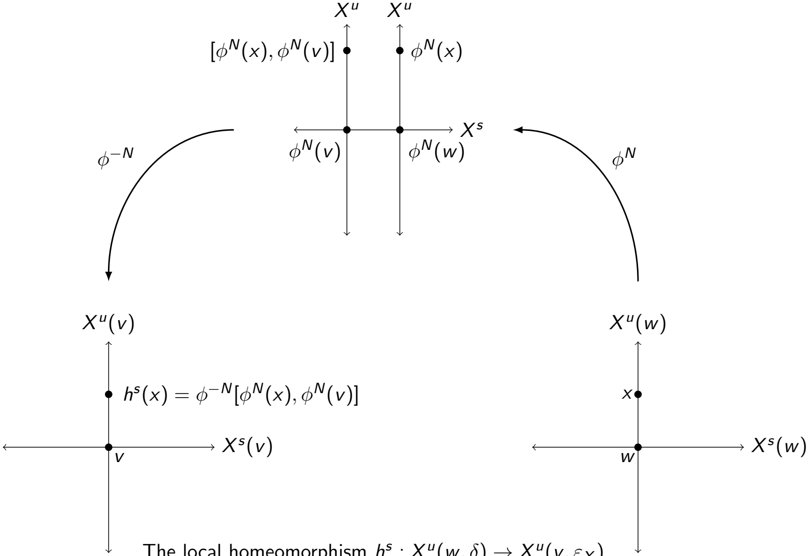
Stable groupoid topology

- Suppose $v \sim_s w$ and $v, w \in X^u(Q)$.
- There exists N such that $d(\varphi^N(v), \varphi^N(w)) < \varepsilon_X/2$.
- Choose $\delta > 0$ small enough so that the diameter of $\varphi^N(X^u(w,\delta))$ and $\varphi^N(X^u(v,\delta))$ is smaller than $\varepsilon_X/2$.

Then there is a local homeomorphism

$$h^s : X^u(w,\delta) \to X^u(v,\delta)$$

which is defined pictorially on the next page.



The local homeomorphism $h^s : X^u(w, \delta) \to X^u(v, \varepsilon_X)$

C^*-algebras

- $C_c(G^s(X, \varphi, P))$ is a complex linear space with
  - Convolution product

$$ab(x, y) = \sum_{(x,z) \in G^s(X,\varphi,P)} a(x, z)b(z, y)$$

  - Involution

$$a^*(x, y) = \overline{a(y, x)}.$$

- Represent $C_c(G^s(X, \varphi, P))$ as bounded operators on $\ell^2(X^h(P))$ via

$$(a\xi)(x) = \sum_{(x,y) \in G^s(X,\varphi,P)} a(x, y)\xi(y).$$

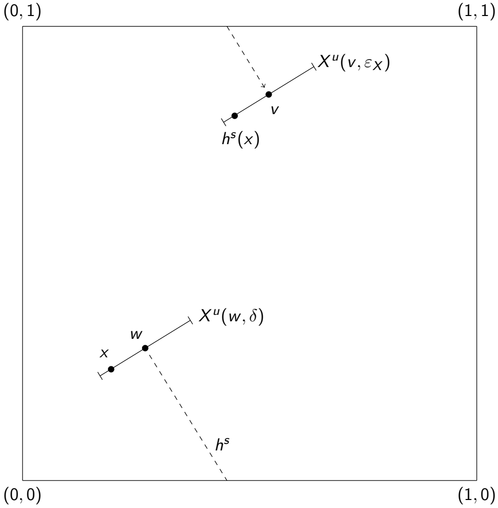
Suppose a is a function in $C_c(G^s(X, \varphi, P))$ with support on the basic set $V^s(v, w, h^s, \delta)$.

Let $\{\delta_x \mid x \in X^h(P)\}$ denote the usual basis of Dirac delta functions in $\mathcal{H} = \ell^2(X^h(P))$.

Then,

$$\pi(a)\delta_x = a(h^s(x), x)\delta_{h^s(x)}$$

if $x \in X^u(w, \delta)$ and zero otherwise.



The local homeomorphism $h^s : X^u(w, \delta) \to X^u(v, \varepsilon_X)$

- Definition: The *stable* C^*-algebra $S(X,\varphi,P)$ is defined to be the completion of $C_c(G^s(X,\varphi,P))$ in the operator norm of $\mathcal{H}$. Moreover, $S(X,\varphi,P)$ is independent of P up to Morita equivalence, so we'll denote it by $\mathcal{S}$ (in fact, they are independent up to isomorphism in the mixing case due to papers by Strung - Deeley and Deeley - Goffeng - Yashinski).

- Definition: The *unstable* C^*-algebra $U(X,\varphi,P)$ is defined to be the completion of $C_c(G^u(X,\varphi,P))$ in the operator norm of $\mathcal{H}$ and is independent of P up to Morita equivalence, so we'll denote it by $\mathcal{U}$. (same).

- Definition: The *homoclinic* C^*-algebra $H(X,\varphi)$ is defined to be the completion of $C_c(G^h(X,\varphi))$ in the operator norm of $\mathcal{H}$ and we denote it by H. (same).

Ruelle Algebras (Putnam)

- The homeomorphism φ induces an inner automorphism on the algebras by

$$\alpha(a)(x,y) = a(\varphi^{-1}(x), \varphi^{-1}(y))$$

- The crossed product

$$R^s := \mathcal{S} \rtimes_\alpha \mathbb{Z},$$

  is called the stable Ruelle algebra.
- Similarly, the unstable and homoclinic Ruelle algebras are the crossed products

$$R^u := \mathcal{U} \rtimes_\alpha \mathbb{Z} \quad \text{and} \quad R^h := H \rtimes_\alpha \mathbb{Z}.$$

Theorem (Putnam and Spielberg 1999)

Suppose (X, φ) is irreducible. Then R^s and R^u are separable, simple, stable, nuclear, purely infinite, and satisfy the UCT. Moreover, they're independent of P up to strong Morita equivalence.

Wieler-Smale spaces

Definition (Wieler 2014)

Suppose V is a compact metric space and $g : V \to V$ is a continuous surjection. We say (V, g) *satisfies Wieler's axioms* if there exists constants $\beta > 0$, $K \in \mathbb{N}_+$, and $\gamma \in (0, 1)$ such that the following hold:

Axiom 1 If $v, w \in V$ satisfy $d(v, w) < \beta$, then

$$d(g^K(v), g^K(w)) \leq \gamma^K d(g^{2K}(v), g^{2K}(w)).$$

Axiom 2 For all $v \in V$ and $\varepsilon \in (0, \beta]$

$$g^K(B(g^K(v), \varepsilon)) \subseteq g^{2K}(B(v, \gamma\varepsilon)).$$

Wieler-Smale spaces

- Suppose V is a compact metric space and $g : V \to V$ is a continuous surjection. We define

$$X_V := \{(v_i)_{i\in\mathbb{N}} \in V^{\mathbb{N}} : \; v_i = g(v_{i+1})\} \tag{1}$$

along with a map $\varphi_g : X_V \to X_V$ given by

$$\varphi_g(v_0, v_1, \ldots) := (g(v_0), v_0, v_1, \ldots). \tag{2}$$

Theorem (Wieler 2014)

(A) *Suppose (V, g) satisfies Wieler's axioms, then (X_V, φ_g) is an irreducible Smale space with totally disconnected stable sets.*

(B) *Suppose (X, φ) is an irreducible Smale space with totally disconnected stable sets. Then there exists a pair (V, g) satisfying Wieler's axioms such that (X, φ) is conjugate to (X_V, φ_g).*

n-solenoids

- Consider $S^1 := \mathbb{R}/\mathbb{Z}$ and $g(x) = nx \pmod 1$.
- We will abuse notation and write nx for $nx \pmod 1$.
- Then (S^1, g) satisfies Wieler's axioms, and Wieler's Theorem gives the Smale space

$$X_{S^1} = \{(x_0, x_1, x_2, \dots) \mid x_i \in [0,1), x_i - nx_{i+1} \in \mathbb{Z}\}$$

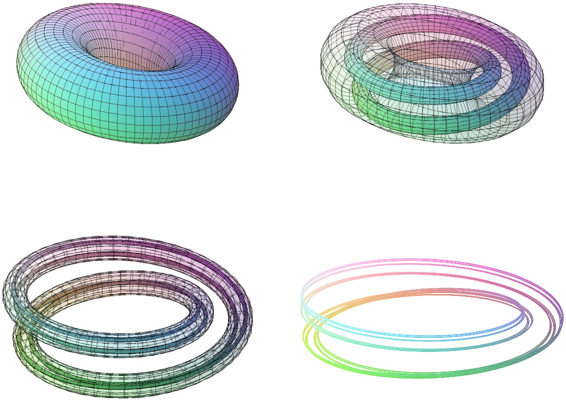
  with dynamics

$$\varphi_g(x_0, x_1, x_2, \dots) = (nx_0, x_0, x_1, x_2, \dots).$$

- For $d(x,y) < 1/2$ let $t = x_0 - y_0$. Then the bracket map is defined by

$$[x, y] := (y_0 + t, y_1 + n^{-1}t, y_2 + n^{-2}t, \dots).$$

A geometric picture of a 2-solenoid



Picture borrowed from:
G. Conner, M. Meilstrup, and D. Repovš, *The geometry and fundamental groups of solenoid complements*, ArXiv: 1212.0128

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