TABULAR ALGEBRAS AND THEIR ASYMPTOTIC VERSIONS

R.M. Green

Department of Mathematics and Statistics
Lancaster University
Lancaster LA1 4YF
England
E-mail: r.m.green@lancaster.ac.uk

Abstract. We introduce tabular algebras, which are simultaneous generalizations of cellular algebras (in the sense of Graham–Lehrer) and table algebras (in the sense of Arad–Blau). We show that if a tabular algebra is equipped with a certain kind of trace map then the algebra has a corresponding asymptotic version whose structure can be explicitly determined. We also study various natural examples of tabular algebras.

To appear in the Journal of Algebra

Introduction

The purpose of this paper is to introduce tabular algebras, a class of associative algebras over $\mathbb{Z}[v, v^{-1}]$. A tabular algebra is defined with a "tabular" basis and an anti-automorphism which are required to satisfy various properties. The construction is a simultaneous generalization of table algebras and of cellular algebras. Our primary objects of study are the tabular bases, and we find that there are important examples of associative algebras in the literature equipped with "natural" bases that turn out to be tabular.

Table algebras were introduced by Arad and Blau [**1**] in order to study irreducible characters and conjugacy classes of finite groups in an abstract setting. Table algebras are related in a precise way to the association schemes of algebraic

2000 *Mathematics Subject Classification.* 16G30, 20C08.
The author was supported in part by an award from the Nuffield Foundation.

combinatorics [**3**] and to Kawada's C-algebras [**17**]. The table algebras in this paper are more general and similar to the discrete hypergroups appearing in the work of Sunder [**29**]; these have applications to subfactors.

Cellular algebras were introduced by Graham and Lehrer [**8**], and are a class of finite dimensional associative algebras defined in terms of a "cell datum" and three axioms. One of their main strengths is that it is relatively straightforward to construct and to classify the irreducible modules for a cellular algebra. Theorem 2.1.1 gives a useful and sufficient criterion for a tabular algebra to be cellular.

There is another—completely different—definition of "cellular algebra" in the literature which is due to Lehman and Weisfeiler [**21**] and which, ironically, is closely related to association schemes. We are not concerned with these algebras here.

We also introduce the notion of a "tabular algebra with trace". In this situation, the tabular algebra is equipped with an **a**-function analogous to the **a**-function appearing in Lusztig's work [**22**, **23**, **24**, **25**] and a trace map which is compatible with this **a**-function in a certain sense which will be made precise. Tabular algebras with trace have some interesting properties which we investigate, such as the existence of a bilinear form (Theorem 2.2.5) that makes transparent the structure of the algebra as a symmetric algebra. We also define asymptotic analogues of tabular algebras by sending the parameter v to ∞ in a controlled way using a general method due to Lusztig [**26**] which makes the structure of the algebra over a suitable field explicit (Theorem 3.2.4).

The second half of the paper is devoted to the detailed study of certain classes of examples of tabular algebras. Our motivation for studying these objects comes from the canonical bases for generalized Temperley–Lieb algebras introduced by the author and J. Losonczy [**15**]. The latter often give, or are closely related to, examples of tabular algebras with trace. There are other interesting examples which we mention in less detail, such as the Hecke algebra of type A, Jones' annular algebra and the Brauer algebra. We conclude with some questions.

1. Preliminaries

In §1, we recall the definitions of table algebras and cellular algebras, and show how they may be used to introduce the notion of a tabular algebra. Table algebras will always be defined over subrings of $\mathbb{C}$, typically $\mathbb{Z}$, and tabular algebras will always be defined over the ring of Laurent polynomials, $\mathbb{Z}[v, v^{-1}]$. For various purposes it is convenient to extend scalars by suitable tensoring; this will usually be made explicit.

1.1 Table algebras.

We begin by defining table algebras, which were introduced in the finite dimensional commutative case by Arad and Blau [**1**].

Definition 1.1.1. A table algebra is a pair $(A, \mathbf{B})$, where A is an associative unital R-algebra for some $\mathbb{Z} \leq R \leq \mathbb{C}$ and $\mathbf{B} = \{b_i : i \in I\}$ is a distinguished basis for A such that $1 \in \mathbf{B}$, satisfying the following three axioms:

(T1) The structure constants of A with respect to the basis $\mathbf{B}$ lie in $\mathbb{R}^+$, the nonnegative real numbers.

(T2) There is an algebra anti-automorphism $^-$ of A whose square is the identity and which has the property that $b_i \in \mathbf{B} \Rightarrow \overline{b_i} \in \mathbf{B}$. (We define $\bar{i}$ by the condition $\overline{b_i} = b_{\bar{i}}$.)

(T3) Let $\kappa(b_i, a)$ be the coefficient of b_i in $a \in A$. Then there is a function $g : \mathbf{B} \times \mathbf{B} \longrightarrow \mathbb{R}^+$ satisfying

$$\kappa(b_m, b_i b_j) = g(b_i, b_m)\kappa(b_i, b_m \overline{b_j}),$$

where $g(b_i, b_m)$ is independent of j, for all i, j, m.

If $R = \mathbb{C}$, it follows from [**1**, Proposition 2.2] that the basis elements of a finite dimensional, commutative table algebra may be uniquely rescaled so that the function g in axiom (T3) sends all pairs of basis elements to $1 \in \mathbb{R}$. This motivates the following

Definition 1.1.2. A *normalized* table algebra $(A, \mathbf{B})$ over R is one whose structure constants lie in $\mathbb{Z}$ and for which the function g in axiom (T3) sends all pairs of basis elements to $1 \in \mathbb{R}$. All table algebras from now on will be normalized.

Remark 1.1.3. Our definition is more general than Arad–Blau's original definition in the sense that we allow A to be noncommutative and/or infinite dimensional. There are many variants of the definition in the literature, such as the generalized table algebras of Arad, Fisman and Muzychuk [**2**], which are not required to be commutative but are of finite rank. A normalized table algebra (in our sense) corresponds to a discrete hypergroup in the sense of Sunder [**29**, Definition IV.1]. The table algebras in this paper are typically finite dimensional and typically commutative.

Definition 1.1.4. If $(A, \mathbf{B})$ is a table algebra and $a \in A$, we write $\text{supp}(a)$ to denote the set of elements of $\mathbf{B}$ which occur with nonzero coefficient in A. (Arad and Blau use the notation $\text{Irr}(a)$.)

A table algebra may be viewed geometrically, as follows.

Lemma 1.1.5. *Let $(A, \mathbf{B})$ be a normalized table algebra over $R \leq \mathbb{R}$ with linear anti-automorphism $\bar{\ }$. There exists a unique positive definite symmetric bilinear form $h : (\mathbb{R} \otimes_R A) \times (\mathbb{R} \otimes_R A) \longrightarrow \mathbb{R}$ with the property that $h(ab, c) = h(a, c\bar{b})$ for all $a, b, c \in A$ and with respect to which the set $\mathbf{B}$ is an orthonormal basis.*

Proof. If $b, b' \in \mathbf{B}$, the orthonormal basis hypothesis requires $h(b, b') = \delta_{b,b'}$, so we define the function h to be the unique (symmetric) bilinear form with this property. The fact that h is positive definite follows easily. To check $h(ab, c) = h(a, c\bar{b})$ for all $a, b, c \in A$, it is enough to consider basis elements a, b, c. However, the definition of f, axioms (T1), (T2), (T3) and the fact that the algebra is normalized imply that

$$h(ab, c) = \kappa(c, ab) = \kappa(a, c\bar{b}) = h(a, c\bar{b}),$$

as required. □

The next result was proved by Arad and Blau [**1**, Proposition 2.5] in the commutative case, and is related to [**2**, Corollary 3.5].

Proposition 1.1.6. *Let $(A, \mathbf{B})$ be a finite dimensional normalized table algebra over R (where $\mathbb{Z} \leq R \leq \mathbb{R}$) with linear anti-automorphism $\bar{\ }$. Then $\mathbb{R} \otimes_R A$ is semisimple as an $\mathbb{R}$-algebra.*

Proof. It suffices to prove that $\mathbb{R} \otimes_R A$ is semisimple as a right module over itself, or that any right ideal in A is complemented. Let K be a right ideal of $\mathbb{R} \otimes_R A$, and let K' be the orthogonal complement of K in $\mathbb{R} \otimes_R A$ with respect to the inner product h of Lemma 1.1.5. Consideration of the equation $h(ab, c) = h(a, c\bar{b})$ where $a \in K$, $c \in K'$ shows that K' is also a right ideal, which completes the proof. □

1.2 Cellular algebras.

Cellular algebras were originally defined by Graham and Lehrer [**8**]. Like table algebras, they are associative algebras with involution which are defined to satisfy certain axioms.

Definition 1.2.1. Let R be a commutative ring with identity. A *cellular algebra* over R is an associative unital algebra, A, together with a cell datum $(\Lambda, M, C, *)$ where:

(C1) Λ is a finite poset. For each $\lambda \in \Lambda$, $M(\lambda)$ is a finite set (the set of "tableaux" of type λ) such that

$$C : \coprod_{\lambda \in \Lambda} (M(\lambda) \times M(\lambda)) \to A$$

is injective with image an R-basis of A.

(C2) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, we write $C(S,T) = C^\lambda_{S,T} \in A$. Then $*$ is an R-linear involutory anti-automorphism of A such that $(C^\lambda_{S,T})^* = C^\lambda_{T,S}$.

(C3) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ then for all $a \in A$ we have

$$a.C^\lambda_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^\lambda_{S',T} \mod A(< \lambda),$$

where $r_a(S', S) \in R$ is independent of T and $A(< \lambda)$ is the R-submodule of A generated by the set

$$\{C^\mu_{S'',T''} : \mu < \lambda, S'' \in M(\mu), T'' \in M(\mu)\}.$$

Remark 1.2.2. We require the poset Λ to be finite. This is not part of the original definition but problems occur in the general theory of cellular algebras (see [**13**, §1.2]) and also in our later results if this hypothesis is omitted.

There are many important examples of algebras in the mathematical literature which turn out to be cellular, some of which we mention later.

1.3 Tabular algebras.

Table algebras and cellular algebras may be (usefully) amalgamated to form "tabular algebras" as follows.

Definition 1.3.1. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. A *tabular algebra* is an $\mathcal{A}$-algebra A, together with a table datum $(\Lambda, \Gamma, B, M, C, *)$ where:

(A1) Λ is a finite poset. For each $\lambda \in \Lambda$, $(\Gamma(\lambda), B(\lambda))$ is a normalized table algebra over $\mathbb{Z}$ and $M(\lambda)$ is a finite set (the set of "tableaux" of type λ). The map

$$C : \coprod_{\lambda \in \Lambda} (M(\lambda) \times B(\lambda) \times M(\lambda)) \to A$$

is injective with image an $\mathcal{A}$-basis of A. We assume that $\mathrm{Im}(C)$ contains a set of mutually orthogonal idempotents $\{1_\varepsilon : \varepsilon \in \mathcal{E}\}$ such that $A = \sum_{\varepsilon, \varepsilon' \in \mathcal{E}} (1_\varepsilon A 1_{\varepsilon'})$ and such that for each $X \in \mathrm{Im}(C)$, we have $X = 1_\varepsilon X 1_{\varepsilon'}$ for some $\varepsilon, \varepsilon' \in \mathcal{E}$. A basis arising in this way is called a *tabular basis.*

(A2) If $\lambda \in \Lambda$, $S, T \in M(\lambda)$ and $b \in B(\lambda)$, we write $C(S, b, T) = C_{S,T}^b \in A$. Then $*$ is an $\mathcal{A}$-linear involutory anti-automorphism of A such that $(C_{S,T}^b)^* = C_{T,S}^{\overline{b}}$, where $\bar{}$ is the table algebra anti-automorphism of $(\Gamma(\lambda), B(\lambda))$. If $g \in \mathbb{C}(v) \otimes_{\mathbb{Z}} \Gamma(\lambda)$ is such that $g = \sum_{b_i \in B(\lambda)} c_i b_i$ for some scalars c_i (possibly involving v), we write $C_{S,T}^g \in \mathbb{C}(v) \otimes_{\mathcal{A}} A$ as shorthand for $\sum_{b_i \in B(\lambda)} c_i C_{S,T}^{b_i}$. We write $\mathbf{c}_\lambda$ for the image under C of $M(\lambda) \times B(\lambda) \times M(\lambda)$.

(A3) If $\lambda \in \Lambda$, $g \in \Gamma(\lambda)$ and $S, T \in M(\lambda)$ then for all $a \in A$ we have

$$a.C_{S,T}^g \equiv \sum_{S' \in M(\lambda)} C_{S',T}^{r_a(S',S)g} \mod A(< \lambda),$$

where $r_a(S', S) \in \Gamma(\lambda)[v, v^{-1}] = \mathcal{A} \otimes_{\mathbb{Z}} \Gamma(\lambda)$ is independent of T and of g and $A(< \lambda)$ is the $\mathcal{A}$-submodule of A generated by the set $\bigcup_{\mu < \lambda} \mathbf{c}_\mu$.

It is an easy consequence of these axioms that any table algebra (with scalars extended to $\mathcal{A}$) is automatically a tabular algebra: set Λ and $M(\lambda)$ to be one-element sets, (Γ, B) to be the table algebra in question, $*$ to be the anti-automorphism of the table algebra and C to be such that $C(m, b, m) = b$. It is also clear that any cellular algebra over $\mathcal{A}$ satisyfing the idempotent conditions in (A1) is a tabular algebra: let $\Lambda, M, C, *$ be as for cellular algebras, and let $(\Gamma(\lambda), B(\lambda))$ be the one-dimensional table algebra spanned by the identity element.

Remark 1.3.2. Note that if we apply $*$ to (A3), we obtain a condition (A3$'$) which reads

$$C_{T,S}^{\overline{g}}.a^* \equiv \sum_{S' \in M(\lambda)} C_{T,S'}^{\overline{g r_a(S',S)}} \mod A(< \lambda).$$

Next, we introduce an **a**-function (in the sense of Lusztig [**22**, **23**, **24**, **25**]) associated to a tabular algebra A.

Definition 1.3.3. Let $g_{X,Y,Z} \in \mathcal{A}$ be one of the structure constants for the tabular basis $\mathrm{Im}(C)$ of A, namely

$$XY = \sum_Z g_{X,Y,Z} Z,$$

where $X, Y, Z \in \mathrm{Im}(C)$. Define, for $Z \in \mathrm{Im}(C)$,

$$\mathbf{a}(Z) = \max_{X,Y \in \mathrm{Im}(C)} \deg(g_{X,Y,Z}),$$

where the degree of a Laurent polynomial is taken to be the highest power of v occurring with nonzero coefficient. We define $\gamma_{X,Y,Z} \in \mathbb{Z}$ to be the coefficient of $v^{\mathbf{a}(Z)}$ in $g_{X,Y,Z}$; this will be zero if the bound is not achieved.

Using the notion of **a**-function, we can now introduce "tabular algebras with trace".

Definition 1.3.4. A *tabular algebra with trace* is a tabular algebra in the sense of Definition 1.3.1 which satisfies the conditions (A4) and (A5) below.

(A4) Let $K = C^b_{S,T}$, $K' = C^{b'}_{U,V}$ and $K'' = C^{b''}_{X,Y}$ lie in $\mathrm{Im}(C)$. Then the maximum bound for $\deg(g_{K,K',K''})$ in Definition 1.3.3 is achieved if and only if $X = S$, $T = U$, $Y = V$ and $b'' \in \mathrm{supp}(bb')$ (see Definition 1.1.4). If these conditions all hold and furthermore $b = b' = b'' = 1$, we require $\gamma_{K,K',K''} = 1$.

(A5) There exists an $\mathcal{A}$-linear function $\tau : A \longrightarrow \mathcal{A}$ (the *tabular trace*), such that $\tau(x) = \tau(x^*)$ for all $x \in A$ and $\tau(xy) = \tau(yx)$ for all $x, y \in A$, that has the property that for every $\lambda \in \Lambda$, $S, T \in M(\lambda)$, $b \in B(\lambda)$ and $X = C^b_{S,T}$, we have

$$\tau(v^{\mathbf{a}(X)} X) = \begin{cases} 1 \mod v^{-1}\mathcal{A}^- & \text{if } S = T \text{ and } b = 1, \\ 0 \mod v^{-1}\mathcal{A}^- & \text{otherwise.} \end{cases}$$

Here, $\mathcal{A}^- := \mathbb{Z}[v^{-1}]$.

We sketch a proof that the Hecke algebra of type A is a tabular algebra with trace.

Example 1.3.5. The Hecke algebra $\mathcal{H} = \mathcal{H}(A_{n-1})$ (over $\mathcal{A}$) of type A_{n-1} is a tabular algebra. The table datum is an extension of the cell datum for $\mathcal{H}$ as a cellular algebra which was given by Graham and Lehrer [**8**, Example 1.2]. In summary, the poset Λ is the set of partitions of n, partially ordered by dominance. The table algebras $(\Gamma(\lambda), B(\lambda))$ are all trivial; that is, $B(\lambda) = \{1\}$ and $\Gamma(\lambda) = \mathbb{Z}$. The set $M(\lambda)$ is the set of standard tableaux of shape λ. Let S, T be standard tableaux of shape λ. The map C takes the triple $(S, 1, T)$ to the Kazhdan–Lusztig basis element C'_w via the Robinson–Schensted correspondence. Since $1 \in \mathrm{Im}(C)$, the idempotent condition is satisfied. The map $*$ sends C'_w to $C'_{w^{-1}}$.

This algebra can be made into a tabular algebra with trace by taking $\tau(h)$ to be the coefficient of T_1 when h is expressed as a linear combination of the basis $\{v^{-\ell(w)} T_w : w \in W(A_{n-1})\}$. (The map τ was defined by Lusztig in [**22**, §1.4].) It follows from the definition of the C'-basis and the properties of Kazhdan–Lusztig polynomials and Duflo involutions (see the introduction to [**23**]) that $\tau(v^{\mathbf{a}(C'_w)} C'_w)$ is $0 \mod v^{-1}\mathcal{A}$ if and only if w is not a Duflo involution. Because all involutions are Duflo involutions in type A, axioms (A4) and (A5) follow from [**23**, Proposition 1.4] and properties of the Robinson–Schensted map.

Example 1.3.5 is trivial from the point of view of table algebras. The next example, although much simpler in structure, is not. We introduce it since it will turn out that all tabular algebras of finite rank with trace, tensored over a suitable field, are isomorphic as abstract algebras to direct sums of algebras of the following form (Theorem 3.2.4).

Example 1.3.6. Let $n \in \mathbb{N}$ and let (Γ, B) be a normalized table algebra over $\mathbb{Z}$. Let $M_n(\mathbb{Z})$ be the ring of all $n \times n$ matrices with integer coefficients. Then the algebra $\mathcal{A} \otimes_{\mathbb{Z}} M_n(\mathbb{Z}) \otimes_{\mathbb{Z}} \Gamma$ is tabular. For the table datum, take Λ to be a one-point set, $(\Gamma(\lambda), B(\lambda)) = (\Gamma, B)$, $M(\lambda) = \{1, 2, \ldots, n\}$, $C(a, b, c) = e_{ac} \otimes b$, where e_{ac} is the usual matrix unit, and $* : e_{ac} \otimes b \mapsto e_{ca} \otimes \bar{b}$. Axioms (A1), (A2), (A3) are easily verified; in this case, the mutually orthogonal idempotents in (A1) are the n elements $e_{aa} \otimes 1$. The algebra can be made into a tabular algebra with trace. In this case, all basis elements have $\mathbf{a}$-value equal to 0. The product of two tabular basis elements can be seen to be zero unless the conditions in axiom (A4) hold, in which case the product of two basis elements is an integral combination of other basis elements parametrized by elements of $\mathrm{supp}(bb')$. For (A5), we note that the map τ which takes $e_{ac} \otimes b$ to 1 if $a = c$ and $b = 1$, and to 0 otherwise, is a trace which satisfies the axiom. (This works because the operation of taking the coefficient of 1 in a normalized table algebra is a trace map; this is a consequence of axiom (T3).)

The point of view taken in this paper is that the main objects of interest are tabular bases, and not the abstract algebras they span. This is in keeping with the philosophy behind table algebras. A finite dimensional normalized table algebra $(A, \mathbf{B})$ over $\mathbb{R}$ is semisimple by Proposition 1.1.6, and therefore of little interest as an abstract algebra A, although its basis $\mathbf{B}$ has many beautiful properties. There may be some merit in considering tabular algebras independently of their bases because, as König and Xi have shown in [**19**, **20**] and other papers, there is a basis-free approach to cellular algebras which leads to interesting results.

2. Properties of tabular algebras

In this section, we investigate some of the consequences of the tabular axioms.

2.1 Relationship with cellular algebras.

It turns out that many naturally occurring tabular algebras are also known examples of cellular algebras. The next result is a sufficient criterion for a tabular algebra to be cellular. This result unifies several proofs already available in the literature.

Theorem 2.1.1. *Let A be a tabular algebra of finite rank with table datum $(\Lambda, \Gamma, B, M, C, *)$; that is $|B(\lambda)| < \infty$ for each $\lambda \in \Lambda$. Suppose that, for some $R \geq \mathbb{Z}$ and for each $\lambda \in \Lambda$, the algebra $R \otimes_{\mathbb{Z}} \Gamma(\lambda)$ is cellular over R with cell datum $(\Lambda_\lambda, M_\lambda, C_\lambda, ^-)$, where $^-$ is the table algebra involution.*

Then $R \otimes_{\mathbb{Z}} A$ is cellular over $R \otimes_{\mathbb{Z}} \mathcal{A}$ with cell datum $(\Lambda', M', C', *)$, where $\Lambda' := \{(\lambda, \lambda') : \lambda \in \Lambda, \lambda' \in \Lambda_\lambda\}$ (ordered lexicographically), $M'((\lambda, \lambda')) := M(\lambda) \times M_\lambda(\lambda')$ and $C((S, s), (T, t))$ (where $(S, s), (T, t) \in M(\lambda) \times M_\lambda(\lambda')$) is equal to $C_{S,T}^{C_\lambda(s,t)}$.

Proof. Axioms (C1) and (C2) follow immediately from the definitions and axioms (A1) and (A2).

To prove axiom (C3), let $\lambda \in \Lambda$ and let $C_\lambda(s, t)$ be a basis element of $\Gamma(\lambda)$ with $s, t \in M_\lambda(\lambda')$. Then by axiom (A3) we have, for any $a \in A$,

$$a.C_{S,T}^{C_\lambda(s,t)} \equiv \sum_{S' \in M(\lambda)} C_{S',T}^{r_a(S',S)C_\lambda(s,t)} \mod A(< \lambda).$$

Since $R \otimes_{\mathbb{Z}} \Gamma(\lambda)$ is cellular over R with cell basis given by C_λ, it follows by axiom (C3) applied to $R \otimes_{\mathbb{Z}} \Gamma(\lambda)$ that

$$r_a(S', S)C_\lambda(s,t) \equiv \sum_{s' \in M_\lambda(\lambda')} r'(S', S, s', s)C_\lambda(s', t) \mod R \otimes \Gamma_\lambda(< \lambda'),$$

where the $r'(S', S, s', s)$ are elements of $R \otimes_{\mathbb{Z}} \mathcal{A}$ which are independent of t (and, by axiom (A3), independent of T). Axiom (C3) follows by tensoring over R. □

A good example of Theorem 2.1.1 concerns the Brauer algebra, which was shown to be cellular by Graham and Lehrer [**8**, §4]. (The reader is referred to their paper for the definition.) Here, we identify the parameter δ with $v + v^{-1}$.

Example 2.1.2. Let $B(n)$ be the Brauer algebra (over $\mathcal{A}$) on n strings. Recall from [**8**, §4] that the algebra has an $\mathcal{A}$-basis consisting of certain triples $[S_1, S_2, w]$ where S_1, S_2 are (arbitrary) involutions on n letters with t fixed points, and w is an element of the symmetric group $\mathcal{S}(t)$ on t letters if $t > 0$, with $w = 1$ if $t = 0$. The algebra has a table datum as follows.

Take Λ to be the set of integers i between 0 and n such that $n - i$ is even, ordered in the natural way. If $\lambda = 0$, take $(\Gamma(\lambda), B(\lambda))$ to be the trivial one-dimensional table algebra; otherwise, take $\Gamma(\lambda)$ to be the group ring $\mathbb{Z}\mathcal{S}(t)$ with basis $B(\lambda) = \mathcal{S}(t)$ and involution $\overline{w} = w^{-1}$. Take $M(\lambda)$ to be the set of involutions on n letters with λ fixed points. Take $C(S_1, w, S_2) = [S_1, S_2, w]$; $\mathrm{Im}(C)$ contains the identity element. The anti-automorphism $*$ sends $[S_1, S_2, w]$ to $[S_2, S_1, w^{-1}]$.

Theorem 2.1.1 is applicable because $\mathcal{S}(t)$ is cellular over $\mathbb{Z}$; this is immediate from setting $v = 1$ in Example 1.3.5. We thus recover Graham–Lehrer's cell datum for the Brauer algebra over $\mathcal{A}$.

A technique similar to that used in Example 2.1.2 may be applied to the case of the partition algebra of [**27**]; again the table algebras are symmetric groups equipped with inversion as the involution. This recovers Xi's main result in [**30**]. The generalized Temperley–Lieb algebra of type H, which is the subject of §5, is also covered by Theorem 2.1.1 (see [**12**, §3.3]); in this case we need R to contain the roots of $x^2 - x - 1$.

Remark 2.1.3. In fact, the algebra $B(n)$ can also be made into a tabular algebra with trace. The trace may be taken to have the property that $\tau(C^w_{S,S})$ is 0 if $w \neq 1$ and is $(v^{-1} + v^{-3})^k$ if $w = 1$ and $k = (n - \lambda)/2$, where $S \in M(\lambda)$.

An interesting non-example of Theorem 2.1.1 which we shall mention again later involves Jones' annular algebra, which we sketch below. We maintain the convention that $\delta = v + v^{-1}$. For the definition of the algebra, see [**16**] and [**8**, §6].

Example 2.1.4. Let J_n be the Jones algebra (over $\mathcal{A}$) on n strings. Recall from [**8**, §6] that the algebra has an $\mathcal{A}$-basis consisting of certain triples $[S_1, S_2, w]$ where S_1, S_2 are certain "annular" involutions on n letters with t fixed points, and w is an element of the cyclic group of order t if $t > 0$, with $w = 1$ if $t = 0$. The algebra has a table datum as follows.

Take Λ to be the set of integers i between 0 and n such that $n - i$ is even, ordered in the natural way. If $\lambda = 0$, take $(\Gamma(\lambda), B(\lambda))$ to be the trivial one-dimensional table algebra; otherwise, take $\Gamma(\lambda)$ to be the group ring over $\mathbb{Z}$ of the cyclic group $\mathbb{Z}_\lambda$, with basis $B(\lambda) = \mathbb{Z}_\lambda$ and involution $\overline{w} = w^{-1}$. Take $M(\lambda)$ to be the set of annular involutions with λ fixed points. Take $C(S_1, w, S_2) = [S_1, S_2, w]$, so that $\text{Im}(C)$ contains the identity element. The anti-automorphism $*$ sends $[S_1, S_2, w]$ to $[S_2, S_1, w^{-1}]$.

Note that the table datum for J_n is considerably simpler than the cell datum for J_n given in [**8**, Theorem 6.15] and that it is defined integrally in terms of a naturally occurring basis. However, we cannot apply Theorem 2.1.1 because group algebras of cyclic groups are generally not cellular with respect to inversion. Graham–Lehrer thus need to use a more complicated involution than $*$ to establish cellularity for J_n. (See [**20**, §§6–7] for more details of the key role played by the involution in the structure of a cellular algebra.)

Remark 2.1.5. The algebra J_n can also be made into a tabular algebra with trace. The trace may be taken to have the property that $\tau(C^w_{S,S})$ is 0 if $w \neq 1$ and is $(v^{-1} + v^{-3})^k$ if $w = 1$ and $k = (n - \lambda)/2$, where $S \in M(\lambda)$. The calculation is similar to the calculations for generalized Temperley–Lieb algebras which we perform in detail later. We shall look at annular involutions more thoroughly in the context of the affine Temperley–Lieb algebra in §6.3.

Two main advantages of considering a finite dimensional cellular algebra as a tabular algebra are the following. First, tabular bases often arise out of natural constructions, such as bases of Kazhdan–Lusztig type, whether or not one is motivated to define tabular algebras. (This also happens for cell bases, but to a lesser extent.) Secondly, there are many examples of cellular algebras which do not have cell bases defined over $\mathbb{Z}[v, v^{-1}]$. Examples of these include Jones' annular algebra above and the generalized Temperley–Lieb algebra of type H [**12**]. In each of these cases, the cell datum relies on the fact that certain polynomials split over the ground ring, although these polynomials do not split over $\mathbb{Z}$. If these algebras are considered as tabular algebras, they can be given natural table data which are defined integrally.

2.2 Structure constants and the tabular trace.

We next study some particular cases of axiom (A3), analogous to the result [**8**, Lemma 1.7].

Definition 2.2.1. Let A be a tabular algebra with table datum $(\Lambda, \Gamma, B, M, C, *)$. Let $\lambda \in \Lambda$ and $S, T, U, V \in M(\lambda)$. We define $\langle T, U\rangle \in \Gamma(\lambda)[v, v^{-1}]$ by the condition

$$C^1_{S,T} C^1_{U,V} \equiv C^{\langle T,U\rangle}_{S,V} \mod A(< \lambda).$$

If $b \in B(\lambda)$, we define $\langle T, U\rangle_b \in \mathcal{A}$ to be the coefficient of b in $\langle T, U\rangle$.

This is well-defined because of the following result.

Lemma 2.2.2. *Maintain the notation of Definition 2.2.1. Then*

$$C^b_{S,T} C^{b'}_{U,V} \equiv C^{b\langle T,U\rangle b'}_{S,V} \mod A(< \lambda)$$

for any $b, b' \in B(\lambda)$, where $\langle T, U\rangle$ is independent of b, b', S and V.

Proof. First consider the expression

$$C^1_{S,T} C^1_{U,V} \equiv C^{\langle T,U\rangle}_{S,V} \mod A(< \lambda).$$

All basis elements $C^b_{S',V'}$ occurring on in the right hand side must satisfy $V' = V$ by (A3) and $S' = S$ by (A3′). Furthermore, $\langle T, U\rangle$ is independent of V by (A3) and independent of S by (A3′). This shows that Definition 2.2.1 is sound.

Starting from Definition 2.2.1, axiom (A3) implies that

$$C^1_{S,T} C^{b'}_{U,V} \equiv C^{\langle T,U\rangle b'}_{S,V} \mod A(< \lambda)$$

for any $b' \in B(\lambda)$ and then axiom (A3′) implies that

$$C^b_{S,T} C^{b'}_{U,V} \equiv C^{b\langle T,U\rangle b'}_{S,V} \mod A(< \lambda)$$

for any $b \in B(\lambda)$, as required. $\square$

Lemma 2.2.3. *Maintain the notation of Definition 2.2.1, with $b \in B(\lambda)$. Denote the degree of the zero Laurent polynomial as $-\infty$ for notational convenience, and suppose the tabular algebra A satisfies axiom (A4).*

(i) *We have* $\deg \langle T, T\rangle_b \leq \deg \mathbf{a}(C^1_{T,T})$, *with equality if and only if* $b = 1$. *Moreover,* $v^{\mathbf{a}(C^1_{T,T})}$ *occurs in* $\langle T, T\rangle_1$ *with coefficient* 1.

(ii) *If* $T \neq U$, *we have* $\deg \langle T, U\rangle_b < \deg \mathbf{a}(C^1_{T,T})$.

Proof. We first prove (i). Consider the expression

$$C^1_{T,T} C^{\overline{b}}_{T,T} \equiv C^{\langle T,T\rangle \overline{b}}_{T,T} \mod A(< \lambda).$$

By axiom (T3), the coefficient of 1 in $\langle T, T\rangle \overline{b}$ is $\langle T, T\rangle_b$, and hence the coefficient of $C^1_{T,T}$ on the right hand side is also $\langle T, T\rangle_b$. By axiom (A4), $C^1_{T,T}$ appears on the right hand side with coefficient of maximal degree if and only if $b = 1$, as $\mathrm{supp}(1\overline{b}) = \{\overline{b}\}$. Furthermore, if $b = 1$, axiom (A4) guarantees that the leading coefficient is 1, as required. Since the structure constants of $\Gamma(\lambda)$ do not involve v, (i) follows.

To prove (ii), we consider the expression

$$C^1_{T,T} C^{\overline{b}}_{U,T} \equiv C^{\langle T,U\rangle \overline{b}}_{T,T} \mod A(< \lambda).$$

Arguing as above, the coefficient of $C^1_{T,T}$ on the right hand side is $\langle T, U\rangle_b$. On the other hand, axiom (A4) shows that $C^1_{T,T}$ does not occur on the right hand side with maximal degree, and (ii) follows. $\square$

The tabular trace has the following key property.

Proposition 2.2.4. *Let A be a (possibly infinite dimensional) tabular algebra with trace. Let $\lambda, \mu \in \Lambda$, $S, T \in M(\lambda)$, $U, V \in M(\mu)$, $b \in B(\lambda)$ and $b' \in B(\mu)$. Then* $\tau(C^b_{S,T} C^{b'}_{U,V}) = 0 \mod v^{-1}\mathcal{A}^-$ *unless* $\lambda = \mu$, $S = V$, $T = U$ *and* $\overline{b} = b'$. *If these conditions hold, then* $\tau(C^b_{S,T} C^{b'}_{U,V}) = 1 \mod v^{-1}\mathcal{A}^-$.

Proof. Set $X := C^b_{S,T}$ and $X' := C^{b'}_{U,V}$. Write

$$C^b_{S,T} C^{b'}_{U,V} = \sum_{X''} g_{X,X',X''} X'',$$

where the sum is taken over tabular basis elements X'' so that the $g_{X,X',X''}$ are structure constants with respect to the tabular basis. Now apply the tabular trace to both sides. By axioms (A4) and (A5), we see that $\tau(g_{X,X',X''}X'')$ lies in $v^{-1}\mathcal{A}^-$ if the $\mathbf{a}$-function bound is not achieved. Even if the bound is achieved, meaning that $T = U$ and hence $\lambda = \mu$, axioms (A4) and (A5) imply that $\tau(g_{X,X',X''}X'')$ will still lie in $v^{-1}\mathcal{A}^-$ unless $S = V$, $X'' = C^1_{S,S}$ and $1 \in \text{supp}(bb')$. Axiom (T3) implies that the last condition happens if and only if $\overline{b} = b'$. If all the conditions hold, we have

$$C^b_{S,T} C^{\overline{b}}_{T,S} \equiv C^{b\langle T,T\rangle\overline{b}}_{S,S} \mod A(< \lambda).$$

By Lemma 2.2.3 (i), the coefficient of $C^1_{S,S}$ in the right hand side is a polynomial whose leading term is $v^{\mathbf{a}(C^1_{T,T})}$. This occurs with coefficient 1 because of the property of $\langle T, T\rangle_1$ described in Lemma 2.2.3 (i) and because the coefficient of $1 \in \Gamma(\lambda)$ in $b\overline{b}$ is 1. Since the degree bound for $g_{X,X',X''}$ is achieved for $X'' = C^1_{S,S}$, we have $\mathbf{a}(C^1_{T,T}) = \mathbf{a}(C^1_{S,S})$. It follows that $\tau(C^b_{S,T}C^{b'}_{U,V}) = 1 \mod v^{-1}\mathcal{A}$, as required. $\square$

Theorem 2.2.5. *Let A be a (possibly infinite dimensional) tabular algebra (over $\mathcal{A}$) with trace τ and table datum $(\Lambda, \Gamma, B, M, C, *)$. Then the map $(x, y) \mapsto \tau(xy^*)$ defines a symmetric, nondegenerate bilinear form on A with the following properties.*

(i) *For all $x, y, z \in A$, $(x, yz) = (xz^*, y)$.*

(ii) *The tabular basis is almost orthonormal with respect to this bilinear form: whenever $X, X' \in \text{Im}(C)$, we have*

$$(X, X') = \begin{cases} 1 \mod v^{-1}\mathcal{A}^- & \textit{if } X = X', \\ 0 \mod v^{-1}\mathcal{A}^- & \textit{otherwise.} \end{cases}$$

Proof. Claim (i) follows from properties of $*$, and claim (ii) is immediate from Proposition 2.2.4. Nondegeneracy follows easily from (ii). Symmetry comes from axiom (A5): $\tau(xy^*) = \tau((xy^*)^*) = \tau(yx^*)$. $\square$

Corollary 2.2.6. *A tabular algebra with trace is a symmetric algebra.*

Proof. Maintain the notation of Theorem 2.2.5. We need to show the existence of a symmetric, associative and nondegenerate bilinear form. The form $f(x, y) := (x, y^*) = \tau(xy)$ is clearly symmetric and associative, and is nondegenerate because $(,)$ is nondegenerate. $\square$

2.3 Properties of the a-function.

The $\mathbf{a}$-function associated to a tabular algebra with trace has similar properties to Lusztig's $\mathbf{a}$-function from [**22**]. We investigate some of these here.

Proposition 2.3.1. *Let A be a tabular algebra with trace with table datum $(\Lambda, \Gamma, B, M, C, *)$. Let $\lambda \in \Lambda$, $b \in B(\lambda)$ and $S, U \in M(\lambda)$. The value of $\mathbf{a}(C^b_{S,U})$ depends only on λ, and not on $S \in M(\lambda)$, $U \in M(\lambda)$ or $b \in B(\lambda)$.*

Note. We may use the notation $\mathbf{a}(\lambda)$ in the sequel, with the obvious meaning.

Proof. By axiom (A4), we know that the coefficient of $C^b_{S,U}$ occurs with maximal degree in the product $C^1_{S,T}C^b_{T,U}$. Now consider the expression

$$\begin{aligned} C^1_{S,T}C^b_{T,U} &\equiv C^{\langle T,T\rangle b}_{S,U} \mod A(<\lambda) \\ &\equiv \langle T,T\rangle_1 C^b_{S,U} + \sum_{1 \neq b' \in B(\lambda)} C^{\langle T,T\rangle_{b'} b'b}_{S,U} \mod A(<\lambda). \end{aligned}$$

By Lemma 2.2.3 (i), the coefficient of $C^b_{S,U}$ occurring in the first term has degree $\mathbf{a}(C^1_{T,T})$. Again by Lemma 2.2.3 (i), the coefficient of $C^b_{S,U}$ in each of the terms of the sum is either zero or has degree strictly less than $\mathbf{a}(C^1_{T,T})$, because each of the terms $\langle T,T\rangle_{b'}$ is either zero or satisfies $\deg \langle T,T\rangle_{b'} < \mathbf{a}(C^1_{T,T})$. (Note that v is not involved in the expansion of $b'b$.) Since $C^b_{S,U}$ occurs with maximal degree, we must have $\mathbf{a}(C^b_{S,U}) = \mathbf{a}(C^1_{T,T})$. The claim follows. □

The next lemma is reminiscent of various results presented in [**23**, §1].

Lemma 2.3.2. *Maintain the usual notation. Let A be a tabular algebra with trace and let K_1, K_2, K_3 be tabular basis elements. Then*

$$\gamma_{K_1,K_2,K_3^*} = \gamma_{K_2,K_3,K_1^*} = \gamma_{K_3,K_1,K_2^*},$$

where γ is as in Definition 1.3.3.

Proof. In order for γ_{K_1,K_2,K_3^*} to be nonzero, K_3^* must appear with maximal degree in the product K_1K_2. If this happens, axiom (A4) requires (among other things) that $K_1 = C^b_{S,T}$, $K_2 = C^{b'}_{T,U}$ and $K_3 = C^{b''}_{U,S}$ for some $\lambda \in \Lambda$, $S, T, U \in M(\lambda)$ and $b, b', b'' \in B(\lambda)$. The same conditions on S, T, U are necessary for $\gamma_{K_2,K_3,K_1^*} \neq 0$ and for $\gamma_{K_3,K_1,K_2^*} \neq 0$, so we can reduce consideration to the case where S, T, U are as above.

Another requirement for $\gamma_{K_1,K_2,K_3^*} \neq 0$, again by axiom (A4), is that $\overline{b''} \in \operatorname{supp}(bb')$. If this condition is met, then $\gamma_{K_1,K_2,K_3^*} \neq 0$; more precisely, $\gamma_{K_1,K_2,K_3^*} = \kappa(\overline{b''}, bb')$ because $\langle T,T\rangle_1$ has leading coefficient 1 by Lemma 2.2.3 (i). Expansion of $bb' \in \Gamma(\lambda)$ and axiom (T3) show that $\kappa(\overline{b''}, bb') = \kappa(1, bb'b'')$. Similar calculations show that $\gamma_{K_2,K_3,K_1^*} = \kappa(1, b'b''b)$ and $\gamma_{K_3,K_1,K_2^*} = \kappa(1, b''bb')$. Since $\kappa(1, xy) = \kappa(1, yx)$ for all $x, y \in \Gamma(\lambda)$, we have

$$\kappa(1, bb'b'') = \kappa(1, b'b''b) = \kappa(1, b''bb'),$$

and the claim follows. □

Corollary 2.3.3. *Let A be a tabular algebra with trace and let $X, Y, Z \in \mathbf{c}_\lambda$, as defined in axiom (A2). Then*

$$\mathbf{a}(Z) = \max_{X,Y \in \mathbf{c}_\lambda} \deg(g_{X,Y,Z}) = \max_{X,Y \in \mathbf{c}_\lambda} \deg(g_{Y,Z,X}) = \max_{X,Y \in \mathbf{c}_\lambda} \deg(g_{Z,X,Y}).$$

Proof. This follows from Proposition 2.3.1, Lemma 2.3.2 and the observation that as X ranges over $\mathbf{c}_\lambda$, so does X^*. □

3. Asymptotic analogues of tabular algebras

Lusztig [**26**] has developed a general method to send the parameter v in an $\mathcal{A}$-algebra to ∞ in a controlled way which leaves the structure of the algebra essentially unchanged. This approach is valid if the algebra satisfies certain properties, which, as we shall see in §3.1, tabular algebras with trace satisfy. This allows us to prove, in §3.2, a result giving the explicit structure of a tabular algebra with trace over a suitable field.

3.1 Cells.

Lusztig's theory is designed to apply to $\mathbb{Q}(v)$-algebras equipped with bases with structure constants lying in $\mathcal{A}$. The basis is assumed to be compatible with a set of mutually orthogonal idempotents as required by axiom (A1).

A slight problem to be overcome is that Lusztig's notion of cells [**26**, §1.3], which differs from the definition of Kazhdan–Lusztig cells [**18**, §1], is rather restrictive. This allows results to be proved for algebras without positivity of structure constants, but is inconvenient here. (All the examples of tabular algebras in this paper have structure constants in $\mathbb{N}[v, v^{-1}]$, although we chose to omit this from the definition.) Instead, we use the definition below; scrutiny of the results in [**26**, §1] shows that they are all still valid, with the same proofs, with this weaker notion of order.

Definition 3.1.1. Let A be a tabular algebra. We introduce a relation, $\preceq$, on the tabular basis by stipulating that $X' \preceq X$ if X' appears with nonzero coefficient in KXK' for some tabular basis elements K, K'.

Remark 3.1.2. If the structure constants of A with respect to the tabular basis lie in $\mathbb{N}[v, v^{-1}]$, which is typical, the relation $\preceq$ is automatically transitive. We could also introduce one-sided (left or right) versions of the relation $\preceq$ which would give rise below to left cells and right cells as in [**18**] or [**5**, Definition 4.1]; we return to this briefly later.

Proposition 3.1.3. *Let A be a tabular algebra with table datum $(\Lambda, \Gamma, B, M, C, *)$ satisfying axiom (A4). Let $\preceq_t$ be the transitive extension of the relation $\preceq$ of Definition 3.1.1. The relation $\sim$ on $\mathrm{Im}(C)$ defined by $Y \sim Z$ if and only if $Y \preceq_t Z$ and $Z \preceq_t Y$ is an equivalence relation. The equivalence classes, known as 2-cells, are parametrized by the elements of Λ, where the class corresponding to λ is $\mathbf{c}_\lambda$.*

Proof. The idempotent condition in axiom (A1) shows that $\sim$ is reflexive.

Let $Y = C^b_{T,U}$ and $Z = C^{b'}_{V,W} \in \mathbf{c}_\lambda$; we will show that $Y \sim Z$. Now

$$YY^* = C^b_{T,U} C^{\overline{b}}_{U,T} \equiv C^{b\langle U,U\rangle \overline{b}}_{T,T} \mod A(< \lambda),$$

which, by Lemma 2.2.3 (i), contains $C^1_{T,T}$ with nonzero coefficient of degree $\mathbf{a}(\lambda)$. There is a similar converse statement: $C^1_{T,T} C^b_{T,U}$ contains Y with nonzero coefficient. This shows that $Y \sim C^1_{T,T}$. Similarly, we have $Z \sim C^1_{V,V}$.

Since $\sim$ is clearly symmetric and transitive, it is an equivalence relation. To finish the proof that $Y \sim Z$, we observe that $C^1_{V,V} \sim C^1_{S,S}$ for any $S, V \in M(\lambda)$. This follows by consideration of the product $C^1_{S,V} C^1_{V,V} C^1_{V,S}$, which shows that $C^1_{S,S} \preceq_t C^1_{V,V}$.

Because Λ is partially ordered, axiom (A3) shows that the equivalence classes of $\sim$ are no bigger than the sets $\mathbf{c}_\lambda$ for fixed λ; these are therefore the equivalence classes. $\square$

The definition of $\mathbf{a}$-function in [**26**] is in terms of 2-cells. Let L_λ be the $\mathcal{A}^-$-span of $\{X : X \in \mathbf{c}_\lambda\}$. Then the $\mathbf{a}$-function $\mathbf{a}(Z)$, where $Z \in \mathbf{c}_\lambda$, is defined by Lusztig to be the smallest nonnegative integer n such that $v^{-n} Z L_\lambda \subseteq L_\lambda$, or ∞ if no such integer exists. We also define A_λ to be the $\mathcal{A}$-submodule of A spanned by $\mathbf{c}_\lambda$. This inherits an associative algebra structure from A in the natural way by setting

$$XX' = \sum_{X'' \in \mathbf{c}_\lambda} g_{X,X',X''} X'',$$

where the $g_{X,X',X''}$ are the structure constants for A.

Lemma 3.1.4. *For a tabular algebra with trace, Lusztig's definition of* **a***-function agrees with Definition 1.3.3.*

Proof. Let $X, Y, Z \in \mathrm{Im}(C)$ and let $\lambda \in \Lambda$ be such that $Z \in \mathbf{c}_\lambda$. We have

$$\mathbf{a}(Z) = \max_{X,Y \in \mathbf{c}_\lambda} \deg(g_{X,Y,Z}) = \max_{X,Y \in \mathbf{c}_\lambda} \deg(g_{Z,X,Y}),$$

where the first equality is by axiom (A4) and the second is by Corollary 2.3.3. The claim follows. □

3.2 Lusztig's properties P_1, P_2 and P_3 and asymptotic tabular algebras.

In order to send the parameter v to ∞ in the correct way, three properties (P_1, P_2 and P_3) are required of a quantum algebra.

Property P_1.

In [**26**, §1.4], a basis B is said to have property P_1 if (a) the **a**-function takes finite values on B and (b) for any 2-cell $\mathbf{c}_\lambda$ and any of the orthogonal idempotents 1_ε, the restriction of $\mathbf{a}$ to $\mathbf{c}_\lambda 1_\varepsilon$ is constant.

Lemma 3.2.1. *Let A be a tabular algebra with trace and with table datum $(\Lambda, \Gamma, B, M, C, *)$. Then* $\mathrm{Im}(C)$ *has property P_1.*

Proof. The two notions of **a**-function agree by Lemma 3.1.4. The **a**-function is constant on 2-cells by Proposition 2.3.1, which proves condition (b) of property P_1. The **a**-function is finite on any given 2-cell because $\langle T, T \rangle \in \mathcal{A}$. □

Following [**26**, §1.4], we write $\widehat{X} := v^{-\mathbf{a}(X)} X$ for any tabular basis element X. The $\mathcal{A}^-$-submodule A_λ^- of A_λ is defined to be generated by the elements $\{\widehat{X} : X \in \mathbf{c}_\lambda\}$. We set t_X to be the image of $\widehat{X}$ in

$$A_\lambda^\infty := \frac{A_\lambda^-}{v^{-1} A_\lambda^-}.$$

The latter is a $\mathbb{Z}$-algebra with basis $\{t_X : X \in \mathbf{c}_\lambda\}$ and structure constants

$$t_X t_{X'} = \sum_{X'' \in \mathbf{c}_\lambda} \gamma_{X,X',X''} t_{X''},$$

where the $\gamma_{X,X',X''} \in \mathbb{Z}$ are as in Definition 1.3.3. We also set

$$A^\infty := \bigoplus_{\lambda \in \Lambda} A_\lambda^\infty;$$

this is a $\mathbb{Z}$-algebra with basis $\{t_X : X \in \mathrm{Im}(C)\}$. It will turn out that, over a suitable field, A^∞ is isomorphic to A.

Property P_2.

In [**26**, §1.5], a basis B with property P_1 is said to have property P_2 if for any 2-cell $\mathbf{c}_\lambda$, the $\mathbb{Z}$-algebra A_λ^∞ admits a generalized unit. This means that there is a subset $\mathcal{D}_\lambda$ of $\mathbf{c}_\lambda$ that has the properties (a) that $t_D t_{D'} = \delta_{D,D'} t''_D$ whenever $D, D' \in \mathcal{D}_\lambda$ and (b) that for any $X \in \mathbf{c}_\lambda$, $t_X \in t_D A_\lambda^\infty t_{D'}$ for some (unique) $D, D' \in \mathcal{D}_\lambda$.

Lemma 3.2.2. *Let A be a tabular algebra with trace and with table datum $(\Lambda, \Gamma, B, M, C, *)$. Then $\mathrm{Im}(C)$ has property P_2.*

Proof. Fix λ. We define $\mathcal{D}_\lambda := \{C^1_{S,S} : S \in M(\lambda)\}$. If $S \neq T \in M(\lambda)$, $D = C^1_{S,S}$ and $D' = C^1_{T,T}$, Lemma 2.2.3 (ii) shows that $t_D t_{D'} = 0$ in A_λ^∞. The last condition of axiom (A4) shows that $t_D t_D = t_D$. This establishes part (a) of property P_2. If $C^b_{S,T} \in \mathbf{c}_\lambda$, we set $D = C^1_{S,S}$ and $D' = C^1_{T,T}$ and then Lemma 2.2.3 (i) gives the existence part (b) of property P_2; uniqueness is by Lemma 2.2.3 (ii). $\square$

Property P_3.

In [**26**, §1.6], it is noted that there is a left A-module structure on A_λ given by

$$X.X' = \sum_{X'' \in \mathbf{c}_\lambda} g_{X,X',X''} X'',$$

where $X \in \mathrm{Im}(C)$ and $X', X'' \in \mathbf{c}_\lambda$. The same formula with $X, X'' \in \mathbf{c}_\lambda$ and $X' \in \mathrm{Im}(C)$ defines a right A-module structure on A_λ. One can then introduce a second indeterminate, v', giving rise to a $\mathbb{Z}[v', v'^{-1}]$-algebra A' which is the same as A except that v is replaced by v'. We define a $\mathbb{Z}[v, v^{-1}, v', v'^{-1}]$-module $\mathfrak{A}_\lambda$ with basis $\{X : X \in \mathbf{c}_\lambda\}$. This is a left A-module and a right A'-module, using the formulae above. Following [**26**, §1.7], we say a basis B has property P_3 if these two structures commute.

Lemma 3.2.3. *Let A be a tabular algebra with trace and with table datum $(\Lambda, \Gamma, B, M, C, *)$. Then* $\mathrm{Im}(C)$ *has property* P_3.

Proof. Let X, X' and X'' be a tabular basis element of A, the basis element of $C^{b'}_{U,V}$ of $\mathfrak{A}_\lambda$, and a tabular basis element of A', respectively. Then, using axioms (A3) and (A3′) and their notation, we obtain

$$\begin{aligned}(X.X').(X''^*) &= \sum_{U' \in M(\lambda)} C^{(r_X(U',U)(v))b'}_{U',V}.(X''^*) \\ &= \sum_{U',V' \in M(\lambda)} C^{(r_X(U',U)(v))b'(\overline{r_{X''}(V',V)(v')})}_{U',V'} \\ &= \sum_{V' \in M(\lambda)} X.C^{b'(\overline{r_{X''}(V',V)(v')})}_{U,V'} \\ &= X.(X'.(X''^*)).\end{aligned}$$

The superscripts in the sums are elements of $\Gamma(\lambda)[v, v^{-1}, v', v'^{-1}]$. The calculation works since the structure constants of $\Gamma(\lambda)$ do not involve v or v'. $\square$

The fact that tabular algebras satisfy Lusztig's properties P_1, P_2 and P_3 gives strong information about their structure. In [**26**, §1.8], Lusztig defines a $\mathbb{Q}(v)$-linear map $\Phi_\lambda : \mathbb{Q}(v) \otimes_{\mathcal{A}} A \longrightarrow \mathbb{Q}(v) \otimes_{\mathbb{Z}} A^\infty_\lambda$ which satisfies

$$\Phi_\lambda(X) = \sum_{D \in \mathcal{D}_\lambda, Z \in \mathbf{c}_\lambda} g_{X,D,Z} t_Z$$

for $X \in \mathrm{Im}(C)$. (There is no assumption that $X \in \mathbf{c}_\lambda$.) By [**26**, Proposition 1.9 (b)], this map is an algebra homomorphism if the three properties are satisfied. We can apply this result to prove the following theorem.

Theorem 3.2.4. *Let A be a tabular algebra of finite rank, with trace and with table datum $(\Lambda, \Gamma, B, M, C, *)$. Let $k := \mathbb{Q}(v)$.*

(i) *For any $\lambda \in \Lambda$, $A_\lambda \cong M_{|M(\lambda)|}(\mathbb{Z}) \otimes_{\mathbb{Z}} \Gamma(\lambda)$ as $\mathbb{Z}$-algebras.*

(ii) *There are k-algebra isomorphisms*

$$k \otimes_{\mathcal{A}} A \cong k \otimes_{\mathbb{Z}} A^\infty \cong \bigoplus_{\lambda \in \Lambda} \left(k \otimes_{\mathbb{Z}} M_{|M(\lambda)|}(\mathbb{Z}) \otimes_{\mathbb{Z}} \Gamma(\lambda)\right).$$

Proof. We first prove (i) by showing that if $X = C^b_{S,T}$, the map sending t_X to $e_{S,T} \otimes b \in M_{|M(\lambda)|}(\mathbb{Z}) \otimes_{\mathbb{Z}} \Gamma(\lambda)$ (where $e_{S,T}$ is a matrix unit) is a ring isomorphism.

Consider a product $t_X t_{X'}$ in A_λ. Unless $X = C^b_{S,T}$ and $X' = C^{b'}_{T,U}$ for some T, the degree bound in XX' is not achieved and $t_X t_{X'}$ will be zero as expected. Otherwise, the tabular basis elements X'' which occur in the product XX' with maximal degree are, by axiom (A4), those of form $C^{b''}_{S,U}$, where $b'' \in \mathrm{supp}(bb')$. Since

$$C^b_{S,T} C^{b'}_{T,U} \equiv C^{b\langle T,T\rangle b'}_{S,U} \mod A(< \lambda),$$

it follows from Lemma 2.3.2 and its proof that $\gamma_{X,X',X''} = \kappa(b'', bb')$. Similarly, the coefficient of $e_{S,U} \otimes b''$ in $(e_{S,T} \otimes b)(e_{T,U} \otimes b')$ is $\kappa(b'', bb')$. This completes the proof of (i).

For the proof of (ii), we note that the isomorphism on the right in the statement follows from (i), so we concentrate on proving $k \otimes_{\mathcal{A}} A \cong k \otimes_{\mathbb{Z}} A^\infty$. Let us define a homomorphism $\Phi : \mathbb{Q}(v) \otimes_{\mathcal{A}} A \longrightarrow \mathbb{Q}(v) \otimes_{\mathbb{Z}} A^\infty$ via the direct sum of the homomorphisms Φ_λ in each component, A^∞_λ.

We claim that Φ is a monomorphism. Let $X \in \mathrm{Im}(C)$, and consider $\widehat{X}$. Let $\lambda \in \Lambda$ (not necessarily such that $X \in \mathbf{c}_\lambda$). Then

$$\Phi_\lambda(\widehat{X}) = \sum_{D \in \mathcal{D}_\lambda, Z \in \mathbf{c}_\lambda} v^{-\mathbf{a}(X)} g_{X,D,Z} t_Z.$$

By axiom (A4), Lemma 3.1.4 and the proof of Lemma 2.3.2, we see that all the terms on the right hand side of this equation lie in $v^{-1}\mathcal{A}^- \otimes_{\mathbb{Z}} A^\infty_\lambda$ unless $X \in \mathbf{c}_\lambda$. If $X \in \mathbf{c}_\lambda$, there is exactly one term on the right hand side for which this is not true, namely the one which corresponds to the unique $D \in \mathcal{D}_\lambda$ with $t_X = t_X t_D$ (as in Property P_2) and $Z = X$. In this case, $\gamma_{X,D,Z} = 1$ and it follows that

$$\Phi_\lambda(\widehat{X}) - t_X \in v^{-1}\mathcal{A}^- \otimes_{\mathbb{Z}} A^\infty_\lambda.$$

Considering all possibilities for $\lambda \in \Lambda$ gives $\Phi(\widehat{X}) - t_X \in v^{-1}\mathcal{A}^- \otimes_{\mathbb{Z}} A^\infty$. Suppose $x \in \ker(\Phi)$. If $x \neq 0$, we may assume without loss of generality that the coefficients of x with respect to the basis $\{\widehat{X} : X \in \mathrm{Im}(C)\}$ lie in $\mathcal{A}^-$, but that not all of them lie in $v^{-1}\mathcal{A}^-$. The statement about $\Phi_\lambda(\widehat{X}) - t_X$ above shows that this cannot happen and thus Φ is a monomorphism. Since A is of finite rank, comparison of dimensions now completes the proof. $\square$

Remark 3.2.5. There is an interesting analogue of Theorem 3.2.4 for tabular algebras of infinite rank, which involves the completion of an $\mathcal{A}^-$-form of the algebra with respect to the v^{-1}-adic topology. We omit the details for reasons of space.

4. Generalized Temperley–Lieb algebras of type ADE

In the remaining sections of the paper, we look in detail at some examples of tabular algebras with trace. We illustrate the results using generalized Temperley–Lieb algebras associated to Hecke algebras of various kinds, starting in §4 with the ADE case. The structure of these algebras is well understood, and their combinatoric properties have been analysed by Fan [**5**] and others. Apart from the details of the tabular trace, most of the work required for the verification of axioms (A1)–(A5) is done in the proofs of Fan's results.

4.1 Definitions.

We start by defining the generalized Temperley–Lieb algebra $TL(X)$; this coincides with the Temperley–Lieb algebra when X is a Coxeter graph of type A_{n-1}.

Definition 4.1.1. Let X be a Coxeter graph of type A_n, D_n or E_n for any $n \in \mathbb{N}$. (We allow the long branch of a graph of type E to be arbitrarily long.) The associative, unital $\mathcal{A}$-algebra $TL(X)$ is defined via generators $b_1, b_2, \ldots b_n$ corresponding to the nodes of the graph and relations

$$b_i^2 = [2]b_i,$$

$$b_i b_j = b_j b_i \quad \text{if nodes } i \text{ and } j \text{ are not connected in } X,$$

$$b_i b_j b_i = b_i \quad \text{if nodes } i \text{ and } j \text{ are connected in } X.$$

As usual, $[2] := v + v^{-1}$.

Let $W(X)$ be the Coxeter group associated to X. A product $w_1 w_2 \cdots w_n$ of elements $w_i \in W(X)$ is called *reduced* if

$$\ell(w_1 w_2 \cdots w_n) = \sum_i \ell(w_i).$$

We reserve the terminology *reduced expression* for reduced products $w_1 w_2 \cdots w_n$ in which every $w_i \in S(X)$.

Call an element $w \in W(X)$ *complex* if it can be written as a reduced product $x_1 w_{ss'} x_2$, where $x_1, x_2 \in W(X)$ and $w_{ss'}$ is the longest element of some rank 2 parabolic subgroup $\langle s, s' \rangle$ such that s and s' do not commute. Denote by $W_c(X)$ the set of all elements of $W(X)$ that are not complex.

For $w \in W_c(X)$, we define $b_w := b_{s_1} b_{s_2} \cdots b_{s_n}$ where $w = s_1 s_2 \cdots s_n$ is a reduced expression for w. This definition does not depend on the choice of reduced expression [**5**, §2.2].

Definition 4.1.2. The set $\{b_w : w \in W_c(X)\}$ is an $\mathcal{A}$-basis for $TL(X)$. We call this the *monomial basis*.

If X is an arbitrary Coxeter graph, the generalized Temperley–Lieb algebra may still be defined in a way which extends Definition 4.1.1. In this case, the algebra $TL(X)$ is the quotient $\mathcal{H}(X)/J(X)$ of the usual $\mathcal{A}$-form of the Hecke algebra $\mathcal{H}(X)$, where $J(X)$ is the two-sided ideal generated by the Kazhdan–Lusztig basis elements $\{C'_w\}$ where w is one of the elements $w_{ss'}$ as above. Such a generalized Temperley–Lieb algebra is equipped with a canonical basis analogous to the Kazhdan–Lusztig basis of the Hecke algebra. The reader is referred to [**15**] for full details.

When X is of type ADE, we may regard the monomial basis as natural in this context as it agrees with canonical basis for $TL(X)$ by [**15**, Theorem 3.6]. The purpose of §4 is to show that $TL(X)$ is a tabular algebra with trace having the monomial basis as its tabular basis.

4.2 Cellular structure and a-function in type ADE.

The following result is well-known and implicit in [**5**].

Proposition 4.2.1. *Let X be a Coxeter graph of type ADE. Then the algebra $TL(X)$ is cellular with cell basis equal to the monomial basis and anti-automorphism given by $* : b_w \mapsto b_{w^{-1}}$ for all $w \in W_c$.*

Proof. The algebras in the statement have been shown to be of finite rank by Graham [**7**, Theorem 7.1]. The decomposition of the monomial basis into cells (i.e., 2-cells in the sense of §3.1) is described explicitly in [**5**]; this provides full details of the poset Λ and the sets $M(\lambda)$. In Fan's terminology, the set Λ is the set of two-sided cells ordered by $\leq_{LR}$ (see [**5**, Definition 4.1]), and $M(\lambda)$ can be identified with the set of involutions in W_c [**5**, Theorem 4.4.4] which belong to the two-sided cell parametrized by λ. This works because a basis element is identifiable from the left cell and the right cell which contain it, by [**5**, Corollary 6.1.4]. This proves axiom (C1). Let $C_{S,T}$ be the basis element in the same left cell as T and the same right cell as S.

Symmetry of the defining relations shows that the map $*$ is an $\mathcal{A}$-linear anti-automorphism, and symmetry of the definition of W_c shows that it permutes the basis elements. Symmetry of the definitions of left and right cells in [**5**, Definition 4.1] shows that $* : C_{S,T} \mapsto C_{T,S}$. This proves axiom (C2).

We can parametrize the basis elements by $C_{S,T}$, where S and T are tableaux of the same shape and correspond to ordered pairs of involutions in the same two-sided cell by the above remarks. The proof of [**5**, Lemma 6.1.1] exhibits, for any $U \in M(\lambda)$, the existence of elements a and a' in $TL(X)$ (depending on T and U but not on S) which have the properties that $C_{S,T}a = C_{S,U}$ and $C_{S,U}a' = C_{S,T}$. Axiom (C3) follows. □

Since the above algebras $TL(X)$ are cellular over $\mathcal{A}$ and the cell basis contains the identity, they are trivially tabular algebras. The interesting aspect from our point of view is that they are naturally tabular algebras with trace.

Lemma 4.2.2. *Let $x, y \in W_c(X)$. Then $b_x b_y = [2]^m b_z$ for some $z \in W_c$. We have $m \leq \min(a(x), a(y)) \leq a(z)$, where a is a certain $\mathbb{N}$-valued function depending only on the two-sided cell $\mathbf{c}_\lambda$ containing x. Furthermore, $m = \min(a(x), a(y))$ if and only if $b_x = C_{S,T}$ and $b_y = C_{T,U}$ for the same T.*

Proof. We take a to be the function defined in [**5**, Definition 2.3.1]. (It will turn out to coincide with our notion of **a**-function, although this is not immediate.) We have $\min(a(x), a(y)) \leq a(z)$ because $a(z) \geq a(x)$ and $a(z) \geq a(y)$ by [**5**, Corollary 4.2.2] and its dual.

The proof of [**5**, Theorem 5.5.1] contains the proof of the other claims, apart from the final statement which is an obvious consequence of the argument given. □

Proposition 4.2.3. *Let X be a Coxeter graph of type ADE. Then the **a**-function (in the sense of Definition 1.3.3) for the monomial basis of $TL(X)$ agrees with Fan's a-function in [**5**] and it satisfies axiom (A4).*

Proof. By [**5**, Proposition 5.4.1], we see that the structure constants of $TL(X)$ are very simple: any two basis elements multiplied together give a power of [2] multiple of another basis element. Consider the equation $b_x b_y = [2]^m b_z$, and suppose that, for fixed z, x and y have been chosen to maximize the number m (meaning that $m = \mathbf{a}(b_z)$). Lemma 4.2.2 now implies that $\mathbf{a}(b_z) \leq a(z)$.

We claim that w and $w' \in W_c(X)$ can be chosen so that $b_w b_{w'} = [2]^{a(z)} b_z$. This will show that $a(z) \leq \mathbf{a}(b_z)$. Write $b_z = C_{S,U}$ for some $S, U \in M(\lambda)$ and some $\lambda \in \Lambda$. Choose T such that $b_t := C_{T,T}$ is a product of commuting generators b_i in the two-sided cell corresponding to λ; this can be arranged by [**5**, Theorem 4.5.1].

Define w and w' such that $b_w = C_{S,T}$ and $b_{w'} = C_{T,U}$, so that

$$b_w b_{w'} = C_{S,T} C_{T,U} \equiv \langle T, T\rangle C_{S,U} \mod A(< \lambda).$$

By [**5**, Lemma 5.2.2], $b_t b_t = [2]^{a(t)} b_t$, and thus $\langle T, T\rangle = [2]^{a(t)} \neq 0$. Since the product of two basis elements is a nonzero multiple of another, this means that $b_w b_{w'} = [2]^{a(t)} b_z$, and that w, w' and z are in the same two-sided cell. Since the a-function is constant on two-sided cells, $a(t) = a(z)$ and $a(z) \leq \mathbf{a}(b_z)$. This proves the first assertion.

Combining these results, we find that the only solutions of $b_x b_y = [2]^m b_z$ for fixed z and $m = \mathbf{a}(z)$ require $m = \min(\mathbf{a}(b_x), \mathbf{a}(b_y)) = \mathbf{a}(b_z) = a(z)$, and therefore $b_x = C_{S,T}$, $b_y = C_{T,U}$ and $b_z = C_{S,U}$. In this case, $[2]^{\mathbf{a}(z)}$ has leading coefficient 1. Axiom (A4) follows. □

4.3 Cell modules and the tabular trace in types ADE.

Recall from [**8**, Definition 2.1] that every cellular algebra possesses a cell module for each $\lambda \in \Lambda$. The parametrization of the irreducible modules of a cellular algebra is in terms of the cell modules. Each cell module has a basis $\{C_S : S \in M(\lambda)\}$ and an action of the algebra with the same structure constants as in axiom (C3).

Proposition 4.3.1. *Let X be a Coxeter graph of type ADE. Then the irreducible modules for $TL(X)$ over the field $\mathbb{Q}(\sqrt{[2]})$ are precisely the cell modules.*

Proof. By [**5**, Theorem 5.6.1], $TL(X)$ is semisimple over $\mathbb{Q}(\sqrt{[2]})$. By [**8**, Theorem 3.8], this means that the cell modules are the irreducible modules, and are pairwise nonisomorphic. □

Definition 4.3.2. Let X be a Coxeter graph of type ADE, and let $TL(X)$ be the associated generalized Temperley–Lieb algebra, with cell datum $(\Lambda, M, C, *)$, as in Proposition 4.2.1. Let τ'_λ be the trace on the irreducible cell module corresponding to $\lambda \in \Lambda$ via Proposition 4.3.1. Then we define the trace τ_λ on A via $\tau_\lambda(x) := v^{-2\mathbf{a}(\lambda)} \tau'_\lambda(x)$. (The $\mathbf{a}$-function is constant on $\mathbf{c}_\lambda$ because it agrees with Fan's a-function by Proposition 4.2.3, and Fan's a-function has this property.)

Remark 4.3.3. Notice that the cell module is defined with an $\mathcal{A}$-basis, so $\tau'_\lambda(x)$ and $\tau_\lambda(x)$ both lie in $\mathcal{A}$ if x is an $\mathcal{A}$-linear combination of cell basis elements. We will use this fact implicitly below without further comment.

Lemma 4.3.4. *Maintain the notation of Definition 4.3.2. Let $Z = b_z$ be an element of the cell basis, and let $\lambda \in \Lambda$ be arbitrary. Then $\tau_\lambda(Z^*) = \tau_\lambda(Z)$ and*

$$\tau_\lambda(v^{\mathbf{a}(Z)} Z) = \begin{cases} 1 \mod v^{-1}\mathcal{A}^- & \text{if } Z = C_{S,S} \text{ for some } S \in M(\lambda), \\ 0 \mod v^{-1}\mathcal{A}^- & \text{otherwise.} \end{cases}$$

Proof. We use the notation of axiom (C3) as applied to cell modules. For any $T \in M(\lambda)$, we have

$$Z.C^\lambda_{S,T} \equiv \sum_{S' \in M(\lambda)} r_Z(S', S) C^\lambda_{S',T} \mod A(< \lambda),$$

which implies that

$$Z.C^\lambda_S = \sum_{S' \in M(\lambda)} r_Z(S', S) C^\lambda_{S'}$$

in the cell module. Applying the trace τ_λ gives

$$\tau_\lambda(v^{\mathbf{a}(Z)} Z) = \sum_{T \in M(\lambda)} v^{-\mathbf{a}(\lambda)} r_Z(T, T).$$

Using axiom (A4), which holds by Proposition 4.2.3, we see that $r_Z(T, T)$ is zero or has degree less than $\mathbf{a}(\lambda)$ unless $Z = C_{T,T}$. In the latter case, $v^{-\mathbf{a}(\lambda)} r_Z(T, T) - 1 \in v^{-1}\mathcal{A}^-$. The second claim follows.

To establish the first claim, we shall use the fact that the $\mathbf{a}$-function is constant on two-sided cells. Note also that if $a \in A(< \lambda)$, we have $\tau_\lambda(a) = 0$.

Suppose $Z = C_{S,U}$, where $S, U \in M(\lambda)$. Let $T \in M(\lambda)$. As in the proof of Proposition 4.2.3, we have $\langle T, T \rangle = [2]^{\mathbf{a}(\lambda)}$, so

$$\begin{aligned} \tau_\lambda([2]^{\mathbf{a}(\lambda)} Z) &= \tau_\lambda(C_{S,T} C_{T,U}) \\ &= \tau_\lambda(C_{T,U} C_{S,T}) \\ &= \langle U, S \rangle \tau_\lambda(C_{T,T}) \\ &= \langle S, U \rangle \tau_\lambda(C_{T,T}) \\ &= \tau_\lambda([2]^{\mathbf{a}(\lambda)} C_{U,S}) \\ &= \tau_\lambda([2]^{\mathbf{a}(\lambda)} Z^*). \end{aligned}$$

(Recall that $\langle , \rangle$ is symmetric by [**8**, Proposition 2.4 (i)].) It follows that $\tau_\lambda(Z) = \tau_\lambda(Z^*)$, as required. □

We can now prove the main result of §4.

Theorem 4.3.5. *Let X be a Coxeter graph of type ADE. Define a trace $\tau : TL(X) \longrightarrow \mathcal{A}$ by $\tau(a) = \sum_{\lambda \in \Lambda} \tau_\lambda(a)$. Then τ satisfies axiom (A5), and this makes $TL(X)$ with its canonical basis into a tabular algebra with trace.*

Proof. It is clear from Lemma 4.3.4 that

$$\tau(v^{\mathbf{a}(Z)} Z) = \begin{cases} 1 \mod v^{-1}\mathcal{A}^- & \text{if } Z = C_{S,S} \text{ for some } S, \\ 0 \mod v^{-1}\mathcal{A}^- & \text{otherwise.} \end{cases}$$

The claim that $\tau(x) = \tau(x^*)$ also follows from Lemma 4.3.4. This establishes axiom (A5) as the condition $b = 1$ is trivially true; the other axioms follow from propositions 4.2.1 and 4.2.3. □

5. Generalized Temperley–Lieb algebras of type H

The generalized Temperley–Lieb algebras of type H_n are an infinite family of finite-dimensional algebras which arise as quotients of (usually infinite dimensional) Hecke algebras associated to Coxeter systems of type H_n. The structure of this algebra was studied in [**12**] by means of a certain basis of diagrams with very convenient properties. It turns out that this basis is natural from other perspectives: one of the main results of [**14**] is that the basis of diagrams agrees with the canonical basis for the algebra in the sense of [**15**]. We show in §5 that this basis has another elegant property, namely that it is the tabular basis of a tabular algebra with trace. Unlike the situation in §4, the table algebras involved here are not trivial.

5.1 The algebra of diagrams.

We recall the definition of $TL(H_n)$, the generalized Temperley–Lieb algebra of type H_n, from [**12**, §1]. This is based on a Coxeter system X of type H_n for $n \geq 2$ whose Coxeter group $W(H_n)$ is given by generating involutions $\{s_i : i \leq n\}$ and defining relations

$$s_i s_j = s_j s_i \quad \text{if } |i - j| > 1,$$

$$s_i s_j s_i = s_j s_i s_j \quad \text{if } |i - j| = 1 \text{ and } \{i, j\} \neq \{1, 2\},$$

$$s_1 s_2 s_1 s_2 s_1 = s_2 s_1 s_2 s_1 s_2.$$

This is an infinite group for $n > 4$. There is a corresponding Hecke algebra with $\mathcal{A}$-basis $\{T_w : w \in W(H_n)\}$ and the usual relations.

The $\mathcal{A}$-algebra $TL(H_n)$ is defined by the monomial basis elements $b_i := b_{s_i}$ as follows:

Definition 5.1.1. Let $n \in \mathbb{N} \geq 2$. We define the associative, unital algebra $TL(H_n)$ over $\mathcal{A}$ via generators $b_1, b_2, \dots b_n$ and relations

$$b_i^2 = [2]b_i,$$

$$b_i b_j = b_j b_i \quad \text{if} \quad |i-j| > 1,$$

$$b_i b_j b_i = b_i \quad \text{if} \quad |i-j| = 1 \quad \text{and} \quad i, j > 1,$$

$$b_i b_j b_i b_j b_i = 3 b_i b_j b_i - b_i \quad \text{if} \quad \{i, j\} = \{1, 2\}.$$

This can also be expressed in terms of the decorated tangles which were defined in [**11**]; for further elaboration and examples, the reader is referred to [**12**, §2].

A tangle is a portion of a knot diagram contained in a rectangle. The tangle is incident with the boundary of the rectangle only on the north and south faces, where it intersects transversely. The intersections in the north (respectively, south) face are numbered consecutively starting with node number 1 at the western (i.e., the leftmost) end. Two tangles are equal if there exists an isotopy of the plane carrying one to the other such that the corresponding faces of the rectangle are preserved setwise. (We call the edges of the rectangular frame "faces" to avoid confusion with the "edges" which are the arcs of the tangle.)

We extend the notion of a tangle so that each arc of the tangle may be assigned a nonnegative integer. If an arc is assigned the value r, we represent this pictorially by decorating the arc with r blobs.

Definition 5.1.2. A decorated tangle is a crossing-free tangle in which each arc is assigned a nonnegative integer. Any arc not exposed to the west face of the rectangular frame must be assigned the integer 0.

The category of decorated tangles, $\mathbb{DT}$, has as its objects the natural numbers (not including zero). The morphisms from n to m are the decorated tangles with n nodes in the north face and m in the south. The source of a morphism is the number of points in the north face of the bounding rectangle, and the target is the number of points in the south face. Composition of morphisms works by concatenation of the tangles, matching the relevant south and north faces together.

Let n be a positive integer. The $\mathcal{A}$-algebra $\mathbb{DT}_n$ has as a free $\mathcal{A}$-basis the morphisms from n to n, where the multiplication is given by the composition in $\mathbb{DT}$.

The edges in a tangle T which connect nodes (i.e., not the loops) may be classified into two kinds: propagating edges, which link a node in the north face with a node in the south face, and non-propagating edges, which link two nodes in the north face or two nodes in the south face.

To explain how this relates to Coxeter systems of type H, we recall from [**12**, §2.2] the notion of an H-admissible diagram.

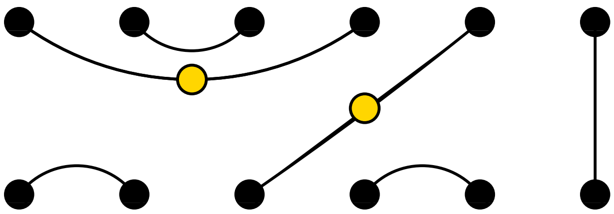
Definition 5.1.3. An H-admissible diagram with n strands is an element of $\mathbb{DT}_n$ with no loops which satisfies the following conditions.

(i) No edge may be decorated if all the edges are propagating.

(ii) If there are non-propagating edges in the diagram, then either there is a decorated edge in the north face connecting nodes 1 and 2, or there is a non-decorated edge in the north face connecting nodes i and $i+1$ for $i > 1$. A similar condition holds for the south face.

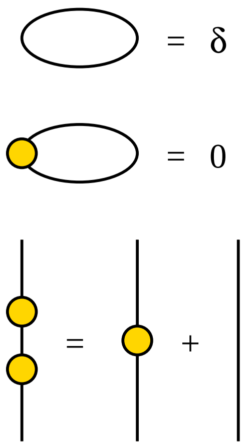
(iii) Each edge carries at most one decoration.

FIGURE 1. An H-admissible diagram for $n = 6$



Definition 5.1.4. The algebra Δ_n (over a commutative ring with identity) has as a basis the H-admissible diagrams with n strands and multiplication induced from that of $\mathbb{DT}_n$ subject to the relations shown in Figure 2.

FIGURE 2. Reduction rules for Δ_n



By [**12**, Lemma 2.2.4], these rules give the free $\mathcal{A}$-module with basis Δ_n the structure of an associative $\mathcal{A}$-algebra; we identify δ with [2]. By [**12**, Theorem 3.4.2], we see that Δ_n is isomorphic as an $\mathcal{A}$-algebra to $TL(H_{n-1})$. The isomorphism is an explicit one which identifies the generators b_i with certain H-admissible diagrams (see [**12**, Proposition 3.1.2]). From now on, we will use this identification implicitly and refer to both algebras as $TL(H_{n-1})$.

5.2 Tabular structure in type H.

The main aim of §5.2 is to show that the diagram basis for $TL(H_n)$ of §5.1 is in fact a tabular basis, making the algebra into a tabular algebra with trace. This is particularly interesting in light of [**14**, Theorem 2.1.3], which shows that the diagram basis is also the canonical basis for $TL(H_n)$ in the sense of [**15**].

In order to understand the trace, it is convenient to make the following definition.

Definition 5.2.1. A trace diagram for $\mathbb{DT}_{n+1}$ consists of an isotopy class of t (possibly zero) non-contractible, non-intersecting loops inscribed on a cylinder, where the ends of the cylinder are labelled "west" and "east". The westmost loop may optionally carry a single decoration, and the integer $0 \leq t \leq n+1$ must be such that $n+1-t$ is even. We denote the set of trace diagrams for $\mathbb{DT}_{n+1}$ by $T(n+1)$, and the free $\mathcal{A}$-module they span by $\mathcal{A}T(n+1)$.

Definition 5.2.2. Let D be a decorated tangle with $n+1$ strands, subject to the reduction rules in Figure 2. Identify the north and south faces of D to form a diagram inscribed on the surface of a cylinder. The reduction rules in Figure 2 are clearly confluent when applied to the cylinder, so the tangle can be reduced to an element, $T(D)$, of $\mathcal{A}T(n+1)$. An element in the support of $T(D)$ consists entirely of a disjoint union of t (possibly zero) non-contractible loops on the cylinder, where the westmost loop may optionally carry a single decoration. (Note that $T(D)$ will be a linear combination of at most two trace diagrams, and will involve a power of δ depending on the number of undecorated loops removed. It is also possible that $T(D) = 0$.) Furthermore, $n+1-t$ is even, because there are $n+1$ strands in the diagram and the reduction rules and isotopy preserve the parity of the number of intersections with a west-east line.

Lemma 5.2.3. *Let $f : T(n+1) \longrightarrow \mathcal{A}$ be any function; extend f by linearity to $\mathcal{A}T(n+1)$. Then there is a trace $\tau_f : TL(H_n) \longrightarrow \mathcal{A}$ defined by $\tau_f(D) := f(T(D))$ and extended linearly.*

Proof. Let D, D' be canonical basis elements. It is clear from Definition 5.2.2 that $T(DD') = T(D'D)$; simply rotate the cylinder about its axis by a half turn. Since $f(T(DD')) = f(T(D'D))$, we see that $\tau_f(DD') = \tau_f(D'D)$. By linearity, $\tau_f(xy) = \tau_f(yx)$, as required. $\square$

Definition 5.2.4. Define $f : T(n+1) \longrightarrow \mathcal{A}$ as follows. If $D \in T(n+1)$ carries a decoration, set $f(D) = 0$. If D is an element of $T(n+1)$ with t non-contractible loops and no decorations, set $f(D) = v^{t-(n+1)}$. (Recall that $t-(n+1)$ is even and nonpositive.)

Let τ be the trace on $TL(H_n)$ given by τ_f for this value of f, as in Lemma 5.2.3.

Theorem 5.2.5. *The algebra $TL(H_n)$ equipped with its canonical basis and the trace τ of Definition 5.2.4 is a tabular algebra with trace.*

Proof. We identify the canonical basis with a set of diagrams in $\mathbb{DT}_{n+1}$ in the usual way. Let Λ be the set of integers t with $0 \leq t \leq n+1$ and $(n+1)-t$ even, ordered in the usual way.

For $\lambda \in \Lambda$, let $(\Gamma(\lambda), B(\lambda))$ be trivial if $\lambda = 0$ or $\lambda = n+1$; for other values of λ, let $\Gamma(\lambda) = \mathbb{Z}[x]/\langle x^2 - x - 1\rangle$ and $B(\lambda) = \{1 + \langle x^2 - x - 1\rangle, x + \langle x^2 - x - 1\rangle\}$. In all cases, the table algebra anti-automorphism is the identity map.

Let $M(\lambda)$ be the set of possible configurations of non-propagating edges in the north face of a canonical basis element with λ propagating edges. The map C produces a basis element from the triple (m, b, m') (where $m, m' \in M(\lambda), b \in B(\lambda)$) as follows. Turn the half-diagram corresponding to m' upside down and place it below the half-diagram corresponding to m. Join any free points in the bottom half to free points in the top half so that they do not intersect. If b is the identity, leave all propagating edges undecorated; if $b = x + \langle x^2 - x - 1\rangle$, decorate the westmost propagating edge.

The map $*$ is the linear extension of top-bottom inversion of the diagrams.

Verification of axioms (A1) to (A3) is straightforward. Notice that the basis contains the identity element.

For axiom (A4), we claim that if $D \in \mathbf{c}_\lambda$, we have $\mathbf{a}(D) = a'(D) := ((n+1)-\lambda)/2$, i.e., the $\mathbf{a}$-function evaluated at a diagram is half the number of non-propagating edges in that diagram. Let $D = C^b_{S,T} \in \mathbf{c}_\lambda$. Direct computation shows that $C^1_{S,S} D = [2]^{a'(D)} C^b_{S,T}$, so $\mathbf{a}(D) \geq a'(D)$. Conversely, the diagram calculus shows that if D' and D'' are canonical basis elements for $TL(H_n)$, the number of loops formed in the product $D'D''$ is bounded above both by $a'(D')$ and $a'(D'')$; this implies that the structure constants appearing in $D'D''$ have degree bounded in the same way. Since D can only appear in a product $D'D''$ if $a'(D') \leq a'(D)$ and $a'(D'') \leq a'(D)$, we have $\mathbf{a}(D) \leq a'(D)$. The claim follows.

The above argument also implies that the only way the $\mathbf{a}$-function bound can be achieved is if the three basis elements D', D'', D concerned come from the same $\mathbf{c}_\lambda$. The fact that loops carrying a single decoration evaluate to zero means that the bound can only be achieved if the pattern of edges at the bottom of D' is the same as the pattern of edges at the top of D''. In this case, we may set $D' = C^b_{S,T}$ and $D'' = C^{b'}_{T,U}$, and properties of the diagram calculus give $D'D'' = [2]^{\mathbf{a}(D)} C^{bb'}_{S,U}$. The assertions of axiom (A4) all follow easily.

Finally, we prove axiom (A5). Consider a basis element D. It is clear by symmetry of the definitions that $\tau(D) = \tau(D^*)$, and thus that $\tau(x) = \tau(x^*)$ for all $x \in A$. To prove the other requirements of the axiom, we note that the diagram corresponding to D has $2r$ non-propagating edges and t propagating edges, where $2r + t = n + 1$ and $r = \mathbf{a}(D)$. To calculate $\tau(D)$, we inscribe D on a cylinder as in Definition 5.2.2. This forms a number r' of contractible loops, where we must have $r' \leq r$ since each loop will include at least one non-propagating edge from each of the north and the south faces of D. In addition, each contractible loop may include an even number of propagating edges from D (e.g. $n = 5$ and $D = b_2 b_3 b_4$). Suppose the total number of propagating edges involved in loops is $2t'$, where $2t' \leq t$. We then have

$$\tau(v^{\mathbf{a}(D)} D) = \tau(v^r D) = c v^{-r-2t'} [2]^{r'}.$$

for some $c \in \mathbb{N}$ (possibly zero). In order for $\tau(v^{\mathbf{a}(D)} D) \not\in v^{-1}\mathcal{A}^-$, we must have $r = r'$ and $t' = 0$. For all these conditions to be met, we need $D = C^b_{S,S}$ for some $S \in M(r)$. If $b \neq 1$, we have $\tau(D) = 0$ by Definition 5.2.4. If $b = 1$, we note that

$$\tau(v^{\mathbf{a}(D)} D) = (1 + v^{-2})^{\mathbf{a}(D)} = 1 \mod v^{-1}\mathcal{A}^{-1}.$$

Axiom (A5) follows, completing the proof. □

6. The affine Temperley–Lieb algebra

The final examples of tabular algebras which we study here are the affine Temperley–Lieb algebras, which turn out to be infinite dimensional tabular algebras. Affine Temperley–Lieb algebras are quotients of affine Hecke algebras which are objects of considerable interest in representation theory. Various people have investigated the representation theory of the affine Temperley–Lieb algebras, for example Martin and Saleur [**28**], the author and K. Erdmann [**10**, **4**] and Graham and Lehrer [**9**].

It follows from the results of [**6**, §4] that the affine Temperley–Lieb algebra has a faithful representation as an algebra of diagrams. By modifying this diagram basis slightly, we obtain a tabular basis which is natural from the viewpoint of Kazhdan–Lusztig theory.

6.1 Diagrams for affine Temperley–Lieb algebras.

We recall the graphical definition of the affine Temperley–Lieb algebras from [**4**, §2.1]. This can be given a more rigorous treatment using categories analogous to our treatment of $TL(H_n)$ in §5.1, but this involves introducing technical definitions which are unnecessary for our purposes. The reader is referred to [**9**, §1] for the categorical approach.

Definition 6.1.1. An affine n-diagram, where $n \in \mathbb{Z}$ satisfies $n \geq 3$, consists of two infinite horizontal rows of nodes lying at the points $\{\mathbb{Z} \times \{0, 1\}\}$ of $\mathbb{R} \times \mathbb{R}$, together with certain curves, called edges, which satisfy the following conditions:

(i) Every node is the endpoint of exactly one edge.

(ii) Any edge lies within the strip $\mathbb{R} \times [0, 1]$.

(iii) If an edge does not link two nodes then it is an infinite horizontal line which does not meet any node. Only finitely many edges are of this type.

(iv) No two edges intersect each other.

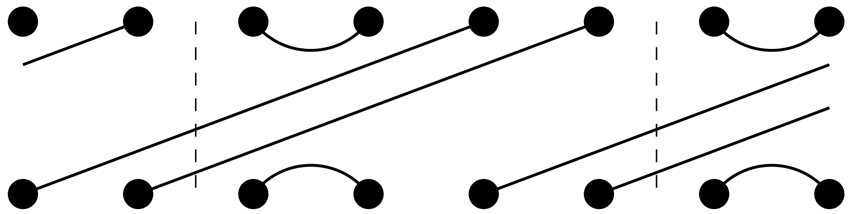
(v) An affine n-diagram must be invariant under shifting to the left or to the right by n.

By an isotopy between diagrams, we mean one which fixes the nodes and for which the intermediate maps are also diagrams which are shift invariant. We will identify any two diagrams which are isotopic to each other, so that we are only interested in the equivalence classes of affine n-diagrams up to isotopy.

Because of the condition (v) in Definition 6.1.1, one can also think of affine n-diagrams as diagrams on the surface of a cylinder, or within an annulus, in a natural way. We shall usually regard the diagrams as diagrams on the surface of a cylinder with n nodes on top and n nodes on the bottom. Under this construction, the top row of nodes becomes a circle of n nodes on one face of the cylinder, which we will refer to as the top circle. Similarly, the bottom circle of the cylinder is the image of the bottom row of nodes. We will use the terms propagating and non-propagating to refer to edges as in Definition 5.1.2.

Example 6.1.2. An example of an affine n-diagram for $n = 4$ is given in Figure 3. The dotted lines denote the periodicity, and should be identified to regard the diagram as inscribed on a cylinder.

FIGURE 3. An affine 4-diagram



Two diagrams, A and B "multiply" in the following way, which was described in [**6**, §4.2]. Put the cylinder for A on top of the cylinder for B and identify all the points in the middle row. This produces a certain (natural) number x of loops. Removal of these loops forms another diagram C satisfying the conditions in Definition 6.1.1. The product AB is then defined to be $[2]^x C$. It is clear that this defines an associative multiplication.

Definition 6.1.3. The associative $\mathcal{A}$-algebra $\mathcal{D}(\widehat{A}_{n-1})$ is the $\mathcal{A}$-linear span of all the affine n-diagrams, with multiplication given as above.

6.2 Generators and relations.

We recall the presentation of $\mathcal{D}(\widehat{A}_{n-1})$ by generators and relations. For further explanation and examples, the reader is referred to [**4**, §2.2].

Definition 6.2.1. Denote by $\bar{i}$ the congruence class of i modulo n, taken from the set $\mathbf{n} := \{1, 2, \dots, n\}$. We index the nodes in the top and bottom circles of each cylinder by these congruence classes in the obvious way.

The diagram u of $\mathcal{D}(\widehat{A}_{n-1})$ is the one satisfying the property that for all $j \in \mathbf{n}$, the point j in the bottom circle is connected to point $\overline{j+1}$ in the top circle by a propagating edge taking the shortest possible route.

The diagram E_i (where $1 \leq i \leq n$) has a horizontal edge of minimal length connecting $\bar{i}$ and $\overline{i+1}$ in each of the circles of the cylinder, and a propagating edge connecting $\bar{j}$ in the top circle to $\bar{j}$ in the bottom circle whenever $\bar{j} \neq \bar{i}, \overline{i+1}$.

Proposition 6.2.2. *The algebra $\mathcal{D}(\widehat{A}_{n-1})$ is generated by elements*

$$E_1, \dots, E_n, u, u^{-1}.$$

It is subject to the following defining relations:

$$E_i^2 = [2]E_i, \tag{1}$$

$$E_i E_j = E_j E_i, \quad \text{if } \bar{i} \neq \overline{j \pm 1}, \tag{2}$$

$$E_i E_{\overline{i \pm 1}} E_i = E_i. \tag{3}$$

$$u E_i u^{-1} = E_{\overline{i+1}}, \tag{4}$$

$$(uE_1)^{n-1} = u^n.(uE_1). \tag{5}$$

Proof. This is [**10**, Proposition 2.3.7]. □

The algebra $\mathcal{D}(\widehat{A}_l)$ is closely related to the generalized Temperley–Lieb algebra of type $\widehat{A}_l$, which is a quotient of the Hecke algebra of type $\widehat{A}_l$ (see [**9**, §0]).

Proposition 6.2.3. *The algebra $TL(\widehat{A}_n)$ is the subalgebra of $\mathcal{D}(\widehat{A}_{n-1})$ spanned by diagrams, D, with the following additional properties:*

(i) *If D has no horizontal edges, then D is the identity diagram, in which point j in the top circle of the cylinder is connected to point j in the bottom circle for all j.*

(ii) *If D has at least one horizontal edge, then the number of intersections of D with the line $x = i + 1/2$ for any integer i is an even number.*

Equivalently, $TL(\widehat{A}_n)$ is the unital subalgebra of $\mathcal{D}(\widehat{A}_{n-1})$ generated by the elements E_i and subject to relations (1)–(3) of Proposition 6.2.2.

Proof. See [**10**, Definition 2.2.1, Proposition 2.2.3]. □

6.3 Annular involutions.

Let D be an affine n-diagram associated to the algebra $\mathcal{D}(\widehat{A}_{n-1})$. Until further notice, we are only concerned with diagrams D with $t > 0$ propagating edges. If $t > 0$, we can define the winding number $w(D)$ as follows.

Definition 6.3.1. Let D be as above. Let $w_1(D)$ be the number of pairs $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ where $i > j$ and $\overline{j}$ in the bottom circle of D is joined to $\overline{i}$ in the top circle of D by an edge which crosses the "seam" $x = 1/2$. We then define $w_2(D)$ similarly but with the condition that $i < j$, and we define $w(D) = w_1(D) - w_2(D)$.

Definition 6.3.2 [**8**, Lemma 6.2]**.** An involution $S \in \mathcal{S}_n$ is annular if and only if, for each pair i, j interchanged by S $(i < j)$, we have

(i) $S[i, j] = [i, j]$ and

(ii) $[i, j] \cap \mathrm{Fix}S = \emptyset$ or $\mathrm{Fix}S \subseteq [i, j]$.

We write $S \in I(t)$ if S has $t > 0$ fixed points, and we write $S \in \mathrm{Ann}(\mathbf{n})$ if S is annular. In case $t = n$ we view the identity permutation as an annular involution.

Using the concept of winding number, we have the following bijection from [**10**, Lemma 3.2.4].

Proposition 6.3.3. *Let D be a diagram for $\mathcal{D}(\widehat{A}_{n-1})$ with t propagating edges $(t > 0)$. Define $S_1, S_2 \in \mathrm{Ann}(\mathbf{n}) \cap I(t)$ and $w \in \mathbb{Z}$ as follows.*

The involution S_1 exchanges points i and j if and only if i is connected to j in the top circle of D. Similarly S_2 exchanges points i and j if and only if i is connected to j in the bottom circle of D. Set $w = w(D)$.

Then this procedure produces a bijection between diagrams D with at least one propagating edge and triples $[S_1, S_2, w]$ as above.

Definition 6.3.4. Suppose $f(v) \in \mathbb{Z}[v, v^{-1}]$ is a Laurent polynomial, and let $S_1, S_2 \in \mathrm{Ann}(\mathbf{n}) \cap I(t)$ for $t > 0$. Then we write $f(S_1, S_2)$ for the element of $\mathcal{D}(\widehat{A}_{n-1})$ obtained from $f(v)$ by linear substitution of $[S_1, S_2, k]$ for v^k.

Now consider the case where $t = 0$, as discussed in [**4**, §7.1].

Definition 6.3.5. Let $I(0)$ be the set of all permutations S of $\mathbb{Z}$ which have the following properties:

(a) for all $k \in \mathbb{Z}$, $S(n + k) = S(k) + n$;

(b) the image of S in $\mathcal{S}_n$ under reduction modulo n is an annular involution with no fixed points.

With this assumption, there is a one to one correspondence between n-diagrams with no propagating edges and triples $[S_1, S_2, k]$ where $S_1, S_2 \in I(0)$ and where $k \geq 0$. Here S_1, S_2 describe the top and the bottom of the diagram as in Proposition 6.3.3, and k is the number of infinite bands. We can extend Definition 6.3.4 to cover this situation as follows.

Definition 6.3.6. Suppose $f(x) \in \mathbb{Z}[x]$ is a polynomial, and let $S_1, S_2 \in I(0)$ (where $I(0)$ is as above). Then we write $f(S_1, S_2)$ for the element of $\mathcal{D}(\widehat{A}_{n-1})$ obtained from $f(x)$ by linear substitution of $[S_1, S_2, k]$ for x^k.

6.4 Tabular structure.

The basis of diagrams for $\mathcal{D}(\widehat{A}_{n-1})$ can be made into a tabular basis after some light modifications (which are trivial if n is odd). The modifications to be made are determined by the properties of Chebyshev polynomials.

Definition 6.4.1. Let $\{U_k(x)\}_{k \in \mathbb{N}}$ be the sequence of polynomials defined by the conditions $U_0(x) = 1$, $U_1(x) = x$ and the recurrence relation $U_{k+1}(x) = xU_k(x) - U_{k-1}(x)$.

Chebyshev polynomials are important in this context due to the following result.

Proposition 6.4.2.

(i) *The algebra $\mathbb{Z}[v, v^{-1}]$ with the involution extending $\bar{v} = v^{-1}$ and basis $\{v^k : k \in \mathbb{Z}\}$ is a normalized table algebra.*

(ii) *The algebra $\mathbb{Z}[x]$ with the trivial involution and basis $\{U_k : k \geq 0\}$ is a normalized table algebra.*

Proof. Part (i) is obvious, because $\mathbb{Z}[v, v^{-1}]$ is the group ring over $\mathbb{Z}$ of the group of integers.

Part (ii) follows from the result

$$U_k(x)U_{k'}(x) = \sum_{i=0}^{k} U_{k'-k+2i}(x),$$

which is valid for $0 \leq k \leq k'$ and which can be established by an easy induction. $\square$

We are now ready to define the table datum for $\mathcal{D}(\widehat{A}_{n-1})$.

Definition 6.4.3. Let $n \geq 3$. Take Λ to be the set of integers i between 0 and n such that $n - i$ is even; order Λ in the usual way. If $\lambda = 0$, take $(\Gamma(\lambda), B(\lambda))$ to be the table algebra $\mathbb{Z}[x]$ of Proposition 6.4.2 (ii). Otherwise, take $(\Gamma(\lambda), B(\lambda))$ to be the table algebra $\mathbb{Z}[v, v^{-1}]$ of Proposition 6.4.2 (i). For $\lambda \neq 0$, set $M(\lambda) := \mathrm{Ann}(\mathbf{n}) \cap I(\lambda)$ as in Definition 6.3.2, and set $M(0) := I(0)$ as in Definition 6.3.5. Take $C(S_1, f, S_2) = f(S_1, S_2)$ as in definitions 6.3.4 and 6.3.6; note that $\mathrm{Im}(C)$ contains the identity element. The anti-automorphism $*$ corresponds to top-bottom reflection of the diagrams.

Note that if n is odd, there is no cell $\lambda = 0$ and $\mathrm{Im}(C)$ above is the same as the diagram basis.

We define the tabular trace for $\mathcal{D}(\widehat{A}_{n-1})$ diagrammatically, following similar lines to the argument in §5.2.

Definition 6.4.4. An affine trace diagram for $\mathcal{D}(\widehat{A}_{n-1})$ consists of an isotopy class of t (possibly zero) non-contractible, non-intersecting loops inscribed on the surface of a torus. We denote the set of affine trace diagrams for $\mathcal{D}(\widehat{A}_{n-1})$ by $\widehat{T}(n)$, and the free $\mathcal{A}$-module they span by $\mathcal{A}\widehat{T}(n)$.

Definition 6.4.5. Let D be an affine n-diagram inscribed on the surface of a cylinder. Identify the top and bottom circles of D to form a diagram inscribed on the surface of a torus. The seam $x = 1/2$ from Definition 6.3.2 maps to a non-contractible circle on the torus which we shall also call the *seam*. The resulting tangle can, by removal of contractible loops, be reduced to one which consists entirely of a disjoint union of t (possibly zero) non-contractible loops on the torus.

There are two kinds of non-contractible loops, which we call regular loops and exceptional loops. Exceptional loops become contractible in the solid torus and arise (for example) from non-contractible loops in the cylinder on which D is inscribed. Regular loops remain non-contractible in the solid torus. The non-intersection criterion guarantees that a tangle arising from this construction cannot have both regular and exceptional loops. If there are no exceptional loops, the integer $n - t$ is even by an argument similar to that given for type H in Definition 5.2.2.

Let $\widehat{T}(D)$ be the element of $\mathcal{A}\widehat{T}(n)$ arising from this construction. Note that $\widehat{T}(D)$ will be a multiple of a single affine trace diagram; the multiple is equal to $[2]^x$ where x is the number of contractible loops removed.

The proof of the following result is similar to that of Lemma 5.2.3, but involves a torus rather than a cylinder.

Lemma 6.4.6. *Let $f : \widehat{T}(n) \longrightarrow \mathcal{A}$ be any function; extend f by linearity to $\mathcal{A}\widehat{T}(n)$. Then there is a trace $\tau_f : \mathcal{D}(\widehat{A}_{n-1}) \longrightarrow A$ defined by $\tau_f(D) := f(\widehat{T}(D))$ and extended linearly.*

Definition 6.4.7. Define $f : \widehat{T}(n) \longrightarrow \mathcal{A}$ by its effect on $f(D)$ for $D \in \widehat{T}(n)$, as follows.

Suppose first that D has no exceptional loops. If all tangles isotopic to $D \in \widehat{T}(n)$ cross the seam of the torus, set $f(D) = 0$. If D is isotopic to an element of $\widehat{T}(n)$ with t contractible loops such that none of the non-contractible loops crosses the seam, set $f(D) = v^{t-n}$. (Recall that in this case, $t - n$ is even and nonpositive.)

Now suppose that D has $k > 0$ exceptional loops. Define $f(D) := v^{-n}\kappa(1, x^k)$, where κ is the function in axiom (T3) applied to the table algebra $\mathbb{Z}[x]$ of Proposition 6.4.2 (ii).

Let τ be the trace on $\mathcal{D}(\widehat{A}_{n-1})$ given by τ_f for this value of f, as in Lemma 6.4.6.

Theorem 6.4.8. *The algebra $\mathcal{D}(\widehat{A}_{n-1})$ equipped with the table datum of Definition 6.4.3 and the trace τ of Definition 6.4.7 is a tabular algebra with trace.*

Proof. The argument is reminiscent of the proof of Theorem 5.2.5, so we only highlight differences.

Note that if $\lambda \neq 0$, $*$ sends $[S_1, S_2, k] \in \mathbf{c}_\lambda$ to $[S_2, S_1, -k]$. However, if $\lambda = 0$, $*$ sends $[S_1, S_2, k] \in \mathbf{c}_\lambda$ to $[S_2, S_1, k]$. Axiom (A2) now follows, and axioms (A1) and (A3) are proved using routine arguments and the diagram calculus.

The $\mathbf{a}$-value of a basis element may be calculated by enumerating the half the number of non-propagating edges in the diagram, not counting the non-contractible bands occurring in elements of $\mathbf{c}_\lambda$ for $\lambda = 0$. Note that if $S \in M(0)$, we have

$$C^b_{S,S} C^{b'}_{S,S} = [2]^{n/2} C^{bb'}_{S,S},$$

where $n/2 = \mathbf{a}(0)$. The argument to prove axiom (A4) is now essentially the same as the one in the proof of Theorem 5.2.5.

The argument to establish axiom (A5) for a basis element $X \in \mathrm{Im}(C)$ also follows easily from the proof of Theorem 5.2.5, except in the case where $X \in \mathbf{c}_0$. If $X \notin \mathbf{c}_0$, we note that if $X = C^b_{S,S}$, then when X is inscribed on a torus as in Definition 6.4.5, the non-propagating edges of X must cross the seam if and only if $b \neq 1$; the proof follows easily from this observation. If, on the other hand, $X = C^b_{S,S}$ and $X \in \mathbf{c}_0$, then $b = U_k(x)$ for some $k \in \mathbb{Z}^+$. The function f of Definition 6.4.7 then satisfies $f(X) = v^{-n}\kappa(1, U_k(x))$, which evaluates to v^{-n} if $k = 0$ and to 0 otherwise. The verification of axiom (A5) is now routine, as $n = 2\mathbf{a}(X)$. □

7. Concluding remarks

The tabular basis in §6 for $\mathcal{D}(\widehat{A}_{n-1})$ is also connected to Kazhdan–Lusztig theory. If $A = \mathcal{D}(\widehat{A}_{n-1})$ and $\mathbf{B}$ is the tabular basis for A, then there exists $\mathbf{B}' \subset \mathbf{B}$ and $A' \leq A$ such that A' is the algebra $TL(\widehat{A}_n)$ of Proposition 6.2.3 and $\mathbf{B}'$ is the canonical basis for $TL(\widehat{A}_n)$. Furthermore, $\mathbf{B}'$ turns out to be the projection of the Kazhdan–Lusztig basis for $\mathcal{H}(\widehat{A}_{n-1})$ to $TL(\widehat{A}_n)$. We call such a pair $(A', \mathbf{B}') \leq (A, \mathbf{B})$ a *sub-tabular algebra*. There are many other natural examples of sub-tabular algebras—for example, the Jones algebra is a sub-tabular algebra of the Brauer algebra (compare examples 2.1.2 and 2.1.4)—and in a subsequent paper we shall consider how generalized Temperley–Lieb algebras of types B and I also fit into this framework.

An interesting question is whether any Hecke algebras of types other than A (recall Example 1.3.5), together with their Kazhdan–Lusztig bases, are examples of tabular (or sub-tabular) algebras. If so, it is plausible that Lusztig's conjectures about Hecke algebras in [**25**, §10] could be interpreted in terms of sub-tabular algebras.

Finally, we note that Graham (unpublished) has investigated Markov traces for generalized Temperley–Lieb algebras which have some elegant properties. It would be interesting to know to what extent these are related to tabular traces.

References

[**1**] Z. Arad and H.I. Blau, *On Table Algebras and Applications to Finite Group Theory*, J. Algebra **138** (1991), 137–185.
[**2**] Z. Arad, E. Fisman and M. Muzychuk, *Generalized table algebras*, Isr. J. Math. **114** (1999), 29–60.
[**3**] E. Bannai and T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin–Cummings, Menlo Park, CA, 1984.
[**4**] K. Erdmann and R.M. Green, *On representations of affine Temperley–Lieb algebras, II*, Pacific J. Math. **191** (1999), 243–273.
[**5**] C.K. Fan, *Structure of a Hecke algebra quotient*, J. Amer. Math. Soc. **10** (1997), 139–167.

[**6**] C.K. Fan and R.M. Green, *On the affine Temperley–Lieb algebras*, Jour. L.M.S. **60** (1999), 366–380.

[**7**] J.J. Graham, *Modular representations of Hecke algebras and related algebras*, Ph.D. thesis, University of Sydney, 1995.

[**8**] J.J. Graham and G.I. Lehrer, *Cellular algebras*, Invent. Math. **123** (1996), 1–34.

[**9**] ———, *The representation theory of affine Temperley–Lieb algebras*, L'Enseignement Math. **44** (1998), 173–218.

[**10**] R.M. Green, *On representations of affine Temperley–Lieb algebras*, Algebras and modules II, CMS Conference Proceedings **24** (1998), 245–261.

[**11**] ———, *Generalized Temperley–Lieb algebras and decorated tangles*, J. Knot Th. Ram. **7** (1998), 155–171.

[**12**] ———, *Cellular algebras arising from Hecke algebras of type H_n*, Math. Zeit. **229** (1998), 365–383.

[**13**] ———, *Completions of cellular algebras*, Comm. Alg. **27** (1999), 5349–5366.

[**14**] ———, *Decorated tangles and canonical bases* (preprint).

[**15**] R.M. Green and J. Losonczy, *Canonical bases for Hecke algebra quotients*, Math. Res. Lett. **6** (1999), 213–222.

[**16**] V.F.R. Jones, *A quotient of the affine Hecke algebra in the Brauer algebra*, L'Enseignement Math. **40** (1994), 313–344.

[**17**] Y. Kawada, *Über den Dualitätsatz der Charaktere nichtcommutativer Gruppen*, Proc. Phys. Math. Soc. Japan **24** (1942), 97–109.

[**18**] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.

[**19**] S. König and C.C. Xi, *On the structure of cellular alghebras*, Algebras and modules II, CMS Conference Proceedings **24** (1998), 365–386.

[**20**] ———, *Cellular algebras: inflations and Morita equivalences*, Jour. L.M.S. **60** (1999), 700–722.

[**21**] A.A. Lehman and B. Yu. Weisfeiler, *Reduction of a graph to a canonical form and an algebra which appears in the process*, NTI Ser. 2 **9** (1968), 12–16.

[**22**] G. Lusztig, *Cells in affine Weyl groups*, Algebraic groups and related topics, Adv. Studies Pure Math 6, North-Holland and Kinokuniya, Tokyo and Amsterdam, 1985, pp. 255–287.

[**23**] ———, *Cells in affine Weyl groups, II*, J. Alg. **109** (1987), 536–548.

[**24**] ———, *Cells in affine Weyl groups, III*, J. Fac. Sci. Tokyo U. (IA) **34** (1987), 223–243.

[**25**] ———, *Cells in affine Weyl groups, IV*, J. Fac. Sci, Tokyo U. (IA) **36** (1989), 297–328.

[**26**] ———, *Quantum groups at $v = \infty$*, Prog. Math. **131** (1995), 199–221.

[**27**] P.P. Martin, *Temperley–Lieb algebras for nonplanar statistical mechanics—the partition algebra construction*, J. Knot Th. Ram. **3** (1994), 51–82.

[**28**] P. Martin and H. Saleur, *On an algebraic approach to higher dimensional statistical mechanics*, Comm. Math. Phys. **158** (1993), 155–190.

[**29**] V.S. Sunder, *II_1 factors, their bimodules and hypergroups*, Trans. Amer. Math. Soc. **330** (1992), 227–256.

[**30**] C.C. Xi, *Partition algebras are cellular*, Compositio Math. **119** (1999), 99–109.