ON PLANAR ALGEBRAS ARISING FROM HYPERGROUPS

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Abstract. Let A be an associative algebra with identity and with trace. We study the family of planar algebras on 1-boxes that arise from A in the work of Jones, but with the added assumption that the labels on the 1-boxes come from a discrete hypergroup in the sense of Sunder. This construction equips the algebra P_n^A with a canonical basis, $\mathbf{B}_n^A$, which turns out to be a "tabular" basis. We examine special cases of this construction to exhibit a close connection between such bases and Kazhdan–Lusztig bases of Hecke algebras of types A, B, H or I.

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Introduction

The purpose of this paper is to exhibit some remarkable links between (i) the planar algebras on 1-boxes arising in the work of Jones [**11**], (ii) tabular algebras, as introduced by the author in [**7**] and (iii) the canonical bases for Hecke algebra quotients that were defined by the author and J. Losonczy in [**8**].

In the work of Jones, a construction is given [**11**, Example 2.2] for a kind of wreath product, P_n^A, of an associative algebra A with the Temperley–Lieb algebra $TL(n, \delta)$. The construction depends on the algebra A having identity and being equipped with a trace function.

Table algebras, which were introduced in [**1**] and generalized in [**2**], are associative algebras with identity equipped with distinguished bases and a natural trace function; the variant we use here is essentially the same as that of Sunder's discrete hypergroups in [**18**]. We are interested here in the "wreath product" of a

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Temperley–Lieb algebra with a table algebra, A. The basis of A equips the resulting algebra with a "canonical" basis. More precisely, Theorem 3.2.3 shows that this new basis is a tabular basis in the sense of [**7**].

We are particularly interested in the case where the table algebra A in question is the Verlinde algebra V_r, which first arose in conformal field theory [**19**]. In this case, the tabular basis of algebra P_n^A is denoted by $\mathbf{B}(n, r)$, where n and r are arbitrary positive integers. The algebra P_n^A has a rich subalgebra structure compatible with the bases $\mathbf{B}(n, r)$; in particular, $\mathbf{B}(n, 1)$ agrees with the usual basis of the Temperley–Lieb algebra and $\mathbf{B}(1, r)$ agrees with the usual basis of the Verlinde algebra. When the parameter n is allowed to vary, this produces examples of Jones' planar algebras, but usually we will not make this explicit and fix n in the results of this paper.

We consider pairs $(P', \mathbf{B}')$ where $\mathbf{B}'$ is a subset of $\mathbf{B}(n, r)$ for some n and r and P' is a subalgebra of P_n^A spanned by $\mathbf{B}'$. What is interesting (Theorem 4.2.5) is that there are important cases of this form where $\mathbf{B}'$ is precisely the canonical basis (in the sense of [**8**]) of a certain quotient of a Hecke algebra of type A, B, H or I. We conclude that there is a close relationship between the Kazhdan–Lusztig bases of [**13**] on the one hand and certain wreath products of discrete hypergroups with Temperley–Lieb algebras on the other.

Although our main results consider the case where the hypergroups of the title are Verlinde algebras, we develop the theory more generally because it is useful in other contexts, such as the recent work of Rui and Xi on cyclotomic Temperley–Lieb algebras [**17**].

Only a very small part of the machinery of planar algebras is necessary for our purposes. In particular, we do not consider C^*-algebras and our main motivation comes from the theory of Hecke algebras of Coxeter groups. We choose to use Jones' formalism because it makes it easy to generalize Martin and Saleur's diagrammatic setting for the blob algebra [**16**] in a rigorous way.

1. Table algebras

In §1, we recall the definition of a table algebra and show that the Verlinde algebra satisfies the table algebra axioms.

1.1 Definitions.

Table algebras were defined by Arad and Blau [**1**] in the finite-dimensional, commutative case. We tend to follow the notation of the paper [**1**], although our definition includes some non-commutative algebras as in [**2**] and, potentially, infinite dimensional algebras. Our algebras are more or less the same as Sunder's discrete hypergroups [**18**, §IV].

Definition 1.1.1. A table algebra is a pair $(A, \mathbf{B})$, where A is an associative unital R-algebra for some $\mathbb{Z} \leq R \leq \mathbb{C}$ and $\mathbf{B} = \{b_i : i \in I\}$ is a distinguished basis for A such that $1 \in \mathbf{B}$, satisfying the following three axioms:

(T1) The structure constants of A with respect to the basis $\mathbf{B}$ lie in $\mathbb{R}^+$, the nonnegative real numbers.

(T2) There is an algebra anti-automorphism $^-$ of A whose square is the identity and that has the property that $b_i \in \mathbf{B} \Rightarrow \overline{b_i} \in \mathbf{B}$. (We define $\bar{i}$ by the condition $\overline{b_i} = b_{\bar{i}}$.)

(T3) Let $\kappa(b_i, a)$ be the coefficient of b_i in $a \in A$. Then there is a function $g : \mathbf{B} \times \mathbf{B} \longrightarrow \mathbb{R}^+$ satisfying

$$\kappa(b_m, b_i b_j) = g(b_i, b_m)\kappa(b_i, b_m\overline{b_j}),$$

where $g(b_i, b_m)$ is independent of j, for all i, j, m.

Following [**7**], we make the following definition which differs slightly from the Arad–Blau notion of a "normalized" table algebra.

Definition 1.1.2. A *normalized* table algebra $(A, \mathbf{B})$ over R is one whose structure constants lie in $\mathbb{Z}$ and for which the function g in axiom (T3) sends all pairs of basis elements to $1 \in \mathbb{R}$. All table algebras from now on will be normalized.

Definition 1.1.3. If $(A, \mathbf{B})$ is a table algebra and $a \in A$, we write $\mathrm{supp}(a)$ to denote the set of elements of $\mathbf{B}$ which occur with nonzero coefficient in a.

Table algebras are equipped with a natural trace function; this is a simple consequence of axiom (T3) (see [**2**, §1]).

Proposition 1.1.4. *Let $(A, \mathbf{B})$ be a normalized table algebra. The linear function t sending $a \in A$ to $\kappa(1, a)$ satisfies $t(xy) = t(yx)$ for all $a \in A$.* $\square$

For our purposes, we shall need to consider certain tensor powers of table algebras. This construction, which is an extension of the external direct product for groups, relies on the following simple fact.

Proposition 1.1.5. *Let $(A_1, \mathbf{B}_1)$ and $(A_2, \mathbf{B}_2)$ be normalized table algebras over $R \leq \mathbb{C}$. Then $(A_1 \otimes A_2, \mathbf{B}_1 \otimes \mathbf{B}_2)$ is a normalized table algebra, where the multiplication on $A_1 \otimes A_2$ is given by the Kronecker product, tensor products are taken over R and the anti-automorphism $^-$ of $A_1 \otimes A_2$ is defined to send $b_1 \otimes b_2$ to $\overline{b_1} \otimes \overline{b_2}$.*

Proof. The identity element of $A_1 \otimes A_2$ is $1 \otimes 1$, which is in the basis. Axioms (T1) and (T2) are immediate.

Consider the coefficient with which a basis element $b_1 \otimes b_2$ occurs in the product $(b_1' \otimes b_2')(b_1'' \otimes b_2'')$. It is clear that

$$\kappa(b_1 \otimes b_2, (b_1' \otimes b_2')(b_1'' \otimes b_2'')) = \kappa(b_1, b_1' b_1'')\kappa(b_2, b_2' b_2'').$$

Since the table algebras $(A_1, \mathbf{B}_1)$ and $(A_2, \mathbf{B}_2)$ are normalized, we have

$$\begin{aligned} \kappa(b_1, b_1' b_1'')\kappa(b_2, b_2' b_2'') &= \kappa(b_1', b_1 \overline{b_1''})\kappa(b_2', b_2 \overline{b_2''}) \\ &= \kappa(b_1' \otimes b_2', (b_1 \otimes b_2)(\overline{b_1''} \otimes \overline{b_2''})). \end{aligned}$$

This proves axiom (T3) and shows that the resulting table algebra is normalized. $\square$

1.2 The Verlinde algebra.

Definition 1.2.1. Let $\{U_n(x)\}_{n \in \mathbb{N}}$ be the sequence of polynomials defined by the conditions $U_0(x) = 1$, $U_1(x) = x$ and the recurrence relation $U_{n+1}(x) = xU_n(x) - U_{n-1}(x)$ for $n > 1$.

The polynomials $U_n(2x)$ are sometimes called "type II Chebyshev polynomials". We use these polynomials to define the Verlinde algebra, which first appeared in [**19**].

Definition 1.2.2. Let $r \geq 1$. The Verlinde algebra, V_r, is defined to be the quotient of $\mathbb{Z}[x]$ by the ideal generated by $U_r(x)$. It has rank r, and we equip it with a $\mathbb{Z}$-basis consisting of the images $u_i(x)$ of the elements $U_i(x)$ for $0 \leq i < r$.

The following result is well-known.

Proposition 1.2.3. *The distinguished basis for V_r given by Definition 1.2.2 gives the Verlinde algebra the structure of a table algebra, where the automorphism $^-$ is the identity map.*

Proof. The structure constants of V_r can be computed by the Clebsch–Gordan rule. Suppose $0 \leq n \leq n' < r$. Then the Clebsch–Gordan rule gives, in our notation,

$$u_n(x)u_{n'}(x) = \sum_{i=0}^{\min(n,r-n'-1)} u_{n'-n+2i}(x).$$

Axioms (T1) and (T2) are now satisfied. Note that $u_0(x)$ occurs with coefficient 1 in this product if $n = n'$, and with coefficient 0 otherwise. Axiom (T3) follows from this observation and [**1**, Lemma 2.1]. □

We conclude this section with some technical lemmas which will be useful later.

Lemma 1.2.4. *Denote by w the element $u_{r-1}(x)$ of V_r. Then, for any $0 \leq i < r$, we have $u_i(x)w = u_{r-1-i}(x)$. In particular, $w^2 = 1$.*

Proof. This is immediate from the formula in the proof of Proposition 1.2.3, with $n' = r - 1$. □

Lemma 1.2.5. *Consider the algebra V_3 over $\mathbb{C}$, with basis $1 = u_0(x)$, $y = u_1(x)$ and $z = u_2(x)$, and the algebra V_2 over $\mathbb{C}$, with basis $\{1', z'\}$. Then there is a homomorphism $\phi : V_3 \longrightarrow V_2$ such that $\phi(1) = 1'$, $\phi(y) = \dfrac{z' + 1'}{\sqrt{2}}$ and $\phi(z) = z'$.*

Proof. We verify that ϕ respects the relations $y^2 = 1 + z$, $yz = zy = y$ and $z^2 = 1$, which presents no problems. □

2. Planar algebras on 1-boxes

In §2, we sketch Jones' construction of planar algebras on 1-boxes arising from associative algebras A, where A has identity and is equipped with a trace map. Although this construction can be made completely rigorous, this would take too much space so we refer the reader to [**11**] for the foundations behind the definitions.

2.1 The Temperley–Lieb algebra.

The concept of a k-box, for $k \in \mathbb{N}$, is defined in [**11**, Definition 1.1], as follows.

Definition 2.1.1. Let k be a nonnegative integer. The *standard k-box*, $\mathcal{B}_k$, is the set $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq k+1,\ 0 \leq y \leq 1\}$, together with the $2k$ marked points

$$1 = (1,1),\ 2 = (2,1),\ 3 = (3,1),\ \dots,\ k = (k,1),$$
$$k+1 = (k,0),\ k+2 = (k-1,0),\ \dots,\ 2k = (1,0).$$

We summarise the definition of the algebra $\mathcal{P}_k(\emptyset)$ from [**11**].

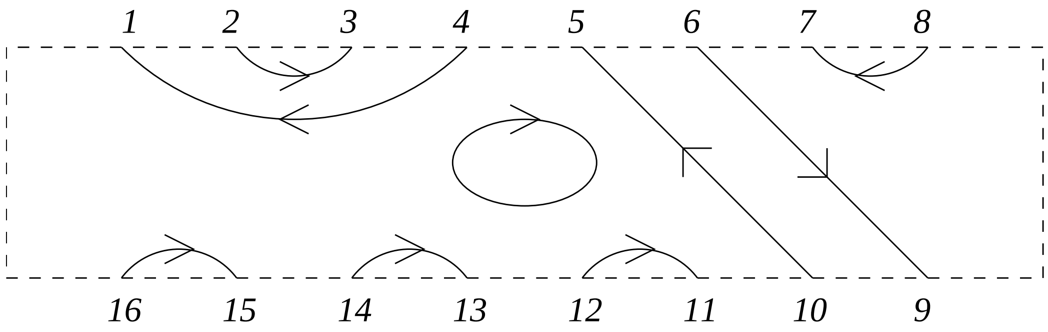
Definition 2.1.2. Let k be a nonnegative integer. An element of $T_k(\emptyset)$ consists of a finite number of oriented disjoint curves (which we usually call "edges"), smoothly embedded in the standard k-box, under smooth orientation-preserving diffeomorphisms of $\mathbb{R}^2$. Curves may be closed (isotopic to circles) but not if their endpoints coincide with marked points of the box. The marked points of the box are endpoints of curves, which meet the box transversely. Otherwise, the curves are disjoint from the box. The orientations of the curves must satisfy the following two conditions.

(i) A curve meeting the r-th marked point of the standard k-box, where r is odd, must exit the box at that point.

(ii) Each connected component of the complement of the union of the curves in the standard k-box may be oriented in such a way that the orientation of a curve coincides with the orientation induced as part of the boundary of the connected component.

Example 2.1.3. Let $k = 8$. An element of $T_8(\emptyset)$ is shown in Figure 1. Note that there are 10 connected components as in Definition 2.1.2 (ii), of which precisely 7 inherit a clockwise orientation.

FIGURE 1. Typical element of $T_8(\emptyset)$



The following definition is a special case of [**11**, Definition 1.8].

Definition 2.1.4. Let k be a nonnegative integer. The associative algebra $\mathcal{P}_k(\emptyset)$ over a commutative ring R with 1 is the free R-module having $T_k(\emptyset)$ as a basis, with multiplication defined as follows. If $T_1, T_2 \in T_k(\emptyset)$, the product T_1T_2 is the element of $T_k(\emptyset)$ obtained by placing T_1 on top of T_2, rescaling vertically by a scalar factor of $1/2$ and applying the appropriate translation to recover a standard k-box.

Note that we may assume that the resulting curves are smooth. The orientations will match up automatically.

The Temperley–Lieb algebra can be easily defined in terms of this formalism, as shown in [**11**, Definition 2.1].

Definition 2.1.5. Let R be a commutative ring with 1. The Temperley–Lieb algebra, $TL(n, \delta)$, is the free $R[\delta]$-module with basis given by the elements of $T_n(\emptyset)$ with no closed loops. The multiplication is inherited from the multiplication on $\mathcal{P}_n(\emptyset)$ except that one multiplies by a factor of δ for each resulting closed loop and then discards the loop.

We usually consider $TL(n, \delta)$ to be an algebra defined over $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$, where $\delta = v + v^{-1}$. The Laurent polynomial $v + v^{-1}$ is often denoted by $[2]$, which will be our preferred notation.

2.2 Planar algebras on 1-boxes.

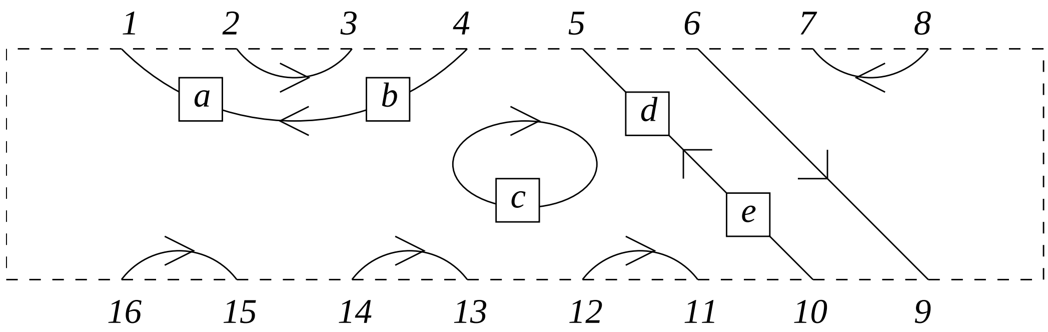
In §2.2, we recall from [**11**, Example 2.2] the construction of the algebra P_n^A from the Temperley–Lieb algebra $TL(n, \delta)$ and the associative R-algebra A, where R is a commutative ring containing δ. The algebra A is assumed to have identity and a trace functional $\mathrm{tr} : A \longrightarrow R$ with $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ and $\mathrm{tr}(1) = \delta$.

Definition 2.2.1. Let A be as above, and let k be a nonnegative integer. We define the tangles $T_k(A)$ to be those that arise from elements of $T_k(\emptyset)$ by adding zero or more 1-boxes labelled by elements of A to each edge. An edge of $T_k(A)$ that is not a loop is called *propagating* if its endpoints have different y-values, and *non-propagating* otherwise. An edge of $T_k(A)$ which is not a loop is called *transitional* if its endpoints lie on different sides of the line $x = 3/2$, and *non-transitional* otherwise. Transitional edges may or may not be propagating.

Figure 2 shows a typical element of $T_8(A)$ which a, b, c, d, e are some elements of the algebra A. There are 2 propagating edges and 6 non-propagating edges.

There are 2 transitional edges (those emerging from points 4 and 16) and 6 non-transitional edges.

FIGURE 2. Typical element of $T_8(A)$



Definitions 2.1.4 and 2.1.5 generalize naturally to this situation, as follows.

Definition 2.2.2. Let k be a nonnegative integer and let A be an R-algebra (as before) with a free R-basis, $\{a_i : i \in I\}$, where $1 \in \{a_i\}$. The associative R-algebra P_k^A is the free R-module having as a basis those elements of $T_k(A)$ satisfying the conditions that

(i) all labels on edges are basis elements a_i,

(ii) each edge has precisely one label and

(iii) there are no closed loops.

The multiplication is defined in the case where T_1 and T_2 are basis elements of P_k^A as above, and extended bilinearly. To calculate the product T_1T_2, place T_1 on top of T_2, rescale vertically by a scalar factor of $1/2$ and apply the appropriate translation to recover a standard k-box. Next, apply relations (a), (b) and (c) below to express the product as an R-linear combination of basis elements, and finally, apply relation (d) below to remove any loops, multiplying by the scalar shown for each loop removed.

FIGURE 3. Relation (a) of Definition 2.2.2

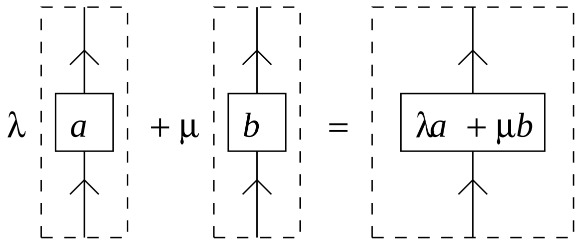


FIGURE 4. Relation (b) of Definition 2.2.2

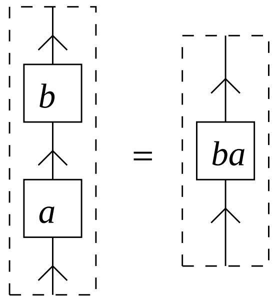


FIGURE 5. Relation (c) of Definition 2.2.2

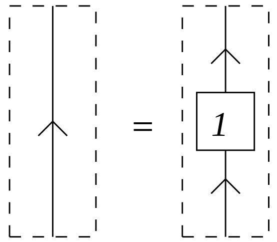
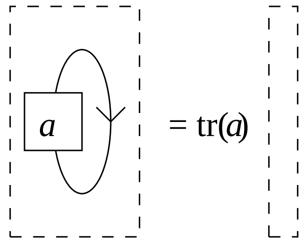


FIGURE 6. Relation (d) of Definition 2.2.2



Remark 2.2.3. The direction on the arrow in relation (d) is immaterial. For a proof that this procedure does define an associative algebra, the reader is referred to [**11**, Example 2.2].

2.3 Planar algebras from table algebras.

If A is a normalized table algebra, we can be more specific about the construction in §2.2.

Definition 2.3.1. Let A be a normalized table algebra over R (containing δ) with distinguished basis $\mathbf{B}$. Then, for $n \in \mathbb{N}$, we define the R-algebra P_n^A to be that arising from Definition 2.2.2 with the basis $\mathbf{B}$ as the distinguished basis; for the trace map, we take $\delta.t$ where t is the trace in Proposition 1.1.4. In this case, we refer to the R-basis of P_n^A that arises in Definition 2.2.2 as the *canonical basis*, $\mathbf{B}_n^A$, of P_n^A.

Note that if A is a one-dimensional unital algebra, this construction yields the Temperley–Lieb algebra (Definition 2.1.5).

For later purposes, we require the following natural anti-automorphism of the algebra P_n^A.

Lemma 2.3.2. *There is an linear anti-automorphism, $*$, of P_n^A permuting the canonical basis. The image, b^*, of a basis element b under this map is obtained by reflecting b in the line $y = 1/2$, reversing the direction of all the arrows and replacing each 1-box labelled by $b_i \in \mathbf{B}$ by a 1-box labelled by $b_{\overline{i}}$.*

Proof. This follows easily from Definition 2.2.2, because the anti-automorphism of the table algebra is $\mathbb{Z}$-linear and the trace of Proposition 1.1.4 satisfies $t(x) = t(x^*)$ for all x. $\square$

Lemma 2.3.3. *Let A be a normalized table algebra as in Definition 2.3.1. Consider a closed loop associated with an element of P_n^A that carries precisely two 1-boxes, each of which is labelled by a basis element. If the labels are b and b', then removal of the loop results in multiplication by zero unless $\overline{b} = b'$, in which case removal of the loop results in multiplication by δ.*

Proof. Let t be the trace map of Proposition 1.1.4. Then $t(bb')$ is 1 if $\overline{b} = b'$ and 0 otherwise. The result follows from Definition 2.3.1. $\square$

As mentioned in [**11**, Example 2.2], the algebra P_n^A is closely related to $A^{\otimes n}$. The following result is a special case of this relationship, and is useful for later purposes.

Proposition 2.3.4. *Let A be a normalized table algebra over R with distinguished basis $\mathbf{B}$. There is an isomorphism ρ of R-algebras from $(A^{\otimes n}, \mathbf{B}^{\otimes n})$ to the subalgebra of P_n^A spanned by all canonical basis elements with no non-propagating edges. The isomorphism takes basis elements to basis elements.*

Proof. Let $b = b_{i_1} \otimes b_{i_2} \otimes \cdots \otimes b_{i_n}$ be a typical basis element from the set $\mathbf{B}^{\otimes n}$. This element is sent by the isomorphism, ρ, to a canonical basis element of P_n^A with no propagating edges, where the decoration on the k-th propagating edge (counting from 1 to n, starting at the left) is b_{i_k} if k is odd, and $b_{\overline{i_k}}$ if k is even. (Note that the k-th propagating edge points upwards if and only if k is odd; this is a consequence of the orientation on the standard n-box.) The relations (a)–(d) of Definition 2.2.2 show that ρ is R-linear and bijective.

We now invoke an idea from Lemma 2.3.2: arrows may be reversed at the expense of applying the table algebra automorphism to their labels. It is now clear that ρ defines an isomorphism of R algebras. $\square$

2.4 The algebra $\mathcal{D}_n^A$.

Let A be a normalized table algebra over R and let P_n^A be the algebra arising from A as in Definition 2.3.1. We now define a certain subalgebra of P_n^A which turns out to be useful for our purposes.

Definition 2.4.1. Let A be a normalized table algebra and let $D \in T_n(A)$ for some n (see Figure 2). Of the connected components of the complement of the union of the curves of D in the standard n-box, there is a unique component which is bounded on the left by the line $x = 0$. We call this the *principal* connected component of D.

Example 2.4.2. In Figure 2, there are precisely 6 edges adjacent to the principal connected component of the tangle shown, including the closed loop.

Definition 2.4.3. Maintain the above notation. A canonical basis element b of P_n^A is said to be *exposed* if the only edges labelled by nonidentity basis elements of A are adjacent to the principal connected component of b. We denote the free R-submodule of P_n^A spanned by the exposed basis elements by $\mathcal{D}_n^A$.

Proposition 2.4.4. *Let A be a normalized table algebra over R and let P_n^A and $\mathcal{D}_n^A$ be as above. Then $\mathcal{D}_n^A$ is a subalgebra of P_n^A.*

Proof. Let T_1 and T_2 be exposed basis elements of P_n^A. It suffices to show that the product T_1T_2 is a linear combination of exposed basis elements. Consider an edge, e, in the product T_1T_2 which is labelled by anything other than a sequence of 1-boxes containing $1 \in A$. Then e must be an extension of an edge of T_i (where $i = 1$ or 2) that is adjacent to the principal connected component of T_i. It follows that e is adjacent to the principal connected component of T_1T_2, and thus that e is exposed. $\square$

Definition 2.4.5. Let n, r be positive integers. Let A be the Verlinde algebra, V_r, with its distinguished basis as in Proposition 1.2.3. We define $\mathcal{D}(n,r) := \mathcal{D}_n^A$ and $P(n,r) := P_n^A$.

The algebra $P(n,r)$ has a useful involution that permutes the basis elements, defined as follows.

Lemma 2.4.6. *Let $\omega : P(n,r) \longrightarrow P(n,r)$ be the linear map defined by sending a canonical basis element D to the element of $P(n,r)$ which differs from D only in that any transitional edges labelled by $b \in V_r$ are relabelled by $u_{r-1}b$. Then ω is an R-algebra automorphism that permutes the canonical basis and preserves $\mathcal{D}(n,r)$ setwise.*

Proof. By Lemma 1.2.4, $u_{r-1}b$ is indeed a basis element of V_r, so ω permutes the canonical basis elements. It is clear that ω is linear and bijective. Also by Lemma 1.2.4, the element $w := u_{r-1}$ satisfies $w^2 = 1$.

Consider an edge or loop, e, in the product DD', and suppose that e is labelled by $g \in V_r$ (where g is not necessarily a basis element). Let e' be the edge or loop corresponding to e in the product $\omega(DD')$, and let its label be $g' \in V_r$. Now e may include either (a) an even or (b) an odd number of transitional edges from $\omega(D)$ and $\omega(D')$. In case (a), e' crosses the line $x = 3/2$ an even number of times and must therefore be a non-transitional edge or a loop; furthermore, $g = g'$ as $w^2 = 1$. In case (b), e crosses the line $x = 3/2$ an odd number of times and must therefore be a transitional edge (and not a loop), and we have $g' = wg$. It follows from these observations that $\omega(DD') = \omega(D)\omega(D')$, and thus that ω is a homomorphism.

To prove the last assertion, note that if D has any (i.e., 2) transitional edges, they must be exposed, as they cross the line $x = 3/2$. This implies that the subalgebra $\mathcal{D}(n,r)$ is invariant under ω. $\square$

Remark 2.4.7. An interesting combinatorial problem arising from these definitions might be to determine an explicit formula for the rank of the algebra $\mathcal{D}(n,r)$ for fixed values of r. For $r = 1$, the rank is the n-th Catalan number, $\frac{1}{n+1}\binom{2n}{n}$; for $r = 2$, the rank is $\binom{2n}{n}$. For $r = 3$, the sequence appears to be related to the number of chains in rooted plane trees (sequence w_3 in [**14**, §2]).

3. Tabular algebras

We recall from [**7**, §1.3] the definition of a tabular algebra. The example to keep in mind throughout is the algebra P_n^A, which will be seen in Theorem 3.2.3 to be a tabular algebra with basis $\mathbf{B}_n^A$ satisfying the five axioms below.

3.1 Tabular algebras and sub-tabular algebras.

Definition 3.1.1. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. A *tabular algebra* is an $\mathcal{A}$-algebra A, together with a table datum $(\Lambda, \Gamma, B, M, C, *)$ where:

(A1) Λ is a finite poset. For each $\lambda \in \Lambda$, $(\Gamma(\lambda), B(\lambda))$ is a normalized table algebra over $\mathbb{Z}$ and $M(\lambda)$ is a finite set. The map

$$C : \coprod_{\lambda \in \Lambda} (M(\lambda) \times B(\lambda) \times M(\lambda)) \to A$$

is injective with image an $\mathcal{A}$-basis of A. We assume that $\mathrm{Im}(C)$ contains a set of mutually orthogonal idempotents $\{1_\varepsilon : \varepsilon \in \mathcal{E}\}$ such that $A = \sum_{\varepsilon, \varepsilon' \in \mathcal{E}} (1_\varepsilon A 1_{\varepsilon'})$ and such that for each $X \in \mathrm{Im}(C)$, we have $X = 1_\varepsilon X 1_{\varepsilon'}$ for some $\varepsilon, \varepsilon' \in \mathcal{E}$. A basis arising in this way is called a *tabular basis.*

(A2) If $\lambda \in \Lambda$, $S, T \in M(\lambda)$ and $b \in B(\lambda)$, we write $C(S, b, T) = C^b_{S,T} \in A$. Then $*$ is an $\mathcal{A}$-linear involutory anti-automorphism of A such that $(C^b_{S,T})^* = C^{\bar{b}}_{T,S}$, where $\bar{\ }$ is the table algebra anti-automorphism of $(\Gamma(\lambda), B(\lambda))$. If $g \in \mathbb{C}(v) \otimes_{\mathbb{Z}} \Gamma(\lambda)$ is such that $g = \sum_{b_i \in B(\lambda)} c_i b_i$ for some scalars c_i (possibly involving v), we write $C^g_{S,T} \in \mathbb{C}(v) \otimes_{\mathcal{A}} A$ as shorthand for $\sum_{b_i \in B(\lambda)} c_i C^{b_i}_{S,T}$. We write $\mathbf{c}_\lambda$ for the image under C of $M(\lambda) \times B(\lambda) \times M(\lambda)$.

(A3) If $\lambda \in \Lambda$, $g \in \Gamma(\lambda)$ and $S, T \in M(\lambda)$ then for all $a \in A$ we have

$$a.C^g_{S,T} \equiv \sum_{S' \in M(\lambda)} C^{r_a(S',S)g}_{S',T} \mod A(< \lambda),$$

where $r_a(S', S) \in \Gamma(\lambda)[v, v^{-1}] = \mathcal{A} \otimes_{\mathbb{Z}} \Gamma(\lambda)$ is independent of T and of g and $A(< \lambda)$ is the $\mathcal{A}$-submodule of A generated by the set $\bigcup_{\mu < \lambda} \mathbf{c}_\mu$.

Next, we recall the **a**-function associated to a tabular algebra A.

Definition 3.1.2. Let $g_{X,Y,Z} \in \mathcal{A}$ be one of the structure constants for the tabular basis $\mathrm{Im}(C)$ of A, namely

$$XY = \sum_Z g_{X,Y,Z} Z,$$

where $X, Y, Z \in \mathrm{Im}(C)$. Define, for $Z \in \mathrm{Im}(C)$,

$$\mathbf{a}(Z) = \max_{X,Y \in \mathrm{Im}(C)} \deg(g_{X,Y,Z}),$$

where the degree of a Laurent polynomial is taken to be the highest power of v occurring with nonzero coefficient. We define $\gamma_{X,Y,Z} \in \mathbb{Z}$ to be the coefficient of $v^{\mathbf{a}(Z)}$ in $g_{X,Y,Z}$; this will be zero if the bound is not achieved.

Using the notion of **a**-function, we recall the definition of "tabular algebras with trace".

Definition 3.1.3. A *tabular algebra with trace* is a tabular algebra in the sense of Definition 3.1.1 that satisfies the conditions (A4) and (A5) below.

(A4) Let $K = C^b_{S,T}$, $K' = C^{b'}_{U,V}$ and $K'' = C^{b''}_{X,Y}$ lie in $\mathrm{Im}(C)$. Then the maximum bound for $\deg(g_{K,K',K''})$ in Definition 3.1.2 is achieved if and only if $X = S$, $T = U$, $Y = V$ and $b'' \in \mathrm{supp}(bb')$ (see Definition 1.1.3). If these conditions all hold and furthermore $b = b' = b'' = 1$, we require $\gamma_{K,K',K''} = 1$.

(A5) There exists an $\mathcal{A}$-linear function $\tau : A \longrightarrow \mathcal{A}$ (the *tabular trace*), such that $\tau(x) = \tau(x^*)$ for all $x \in A$ and $\tau(xy) = \tau(yx)$ for all $x, y \in A$, that has the property that for every $\lambda \in \Lambda$, $S, T \in M(\lambda)$, $b \in B(\lambda)$ and $X = C^b_{S,T}$, we have

$$\tau(v^{\mathbf{a}(X)} X) = \begin{cases} 1 \mod v^{-1}\mathcal{A}^- & \text{if } S = T \text{ and } b = 1, \\ 0 \mod v^{-1}\mathcal{A}^- & \text{otherwise.} \end{cases}$$

Here, $\mathcal{A}^- := \mathbb{Z}[v^{-1}]$.

The main results of this paper may be described in terms of "sub-tabular algebras", which we now introduce.

Definition 3.1.4. Let $(A, \mathbf{B})$ be a tabular algebra A together with its tabular basis $\mathbf{B} = \mathrm{Im}(C)$. Then a sub-tabular algebra is a pair $(A', \mathbf{B}')$, where $\mathbf{B}' \subseteq \mathbf{B}$ and A' is a subalgebra of A with $\mathbf{B}'$ as a basis.

Note that a sub-tabular algebra may or may not be a tabular algebra in its own right. Some, but not all, of the examples of sub-tabular algebras that we consider in §4.2 are also tabular algebras.

3.2 Tabular structure of P_n^A.

In §3.2, we suppose $(A, \mathbf{B})$ is a table algebra and we show that P_n^A, equipped with its canonical basis $\mathbf{B}_n^A$, is a tabular algebra in a natural way. This involves equipping the algebra P_n^A with a certain trace, which is easily done using the formalism of planar algebras.

Recall from [**11**, Definition 1.2.8] that for a spherical planar algebra (i.e., one for which relations such as that in Figure 6 may be transformed into their mirror images) we may define a trace on a k-box, x, by the following procedure. First, for each $1 \leq i \leq k$, join the point i to the point $2k+1-i$ using suitably oriented non-intersecting curves, and then repeatedly apply relation (d) of Definition 2.2.2 (with respect to a suitable trace on the underlying table algebra) to obtain $\mathrm{tr}(x)$.

Definition 3.2.1. Let $(A, \mathbf{B})$ be a table algebra over $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ and let $\delta = v + v^{-1}$. We define the trace $\tau : P_n^A \longrightarrow R$ by $\tau(x) := v^{-n}\mathrm{tr}(x)$, where $\mathrm{tr} = \mathrm{tr}_L = \mathrm{tr}_R$ is Jones' trace from [**11**, Definition 1.28] compatible with the trace $\delta.t$ on A, with t as in Proposition 1.1.4.

Example 3.2.2. The identity element $1 \in P_n^A$ satisfies $\mathrm{tr}(1) = \delta^n$ and $\tau(1) = (1 + v^{-2})^n$.

The element x of $P_8(A)$ shown in Figure 2 satisfies $\mathrm{tr}(x) = \delta^3 t(ab)t(c)t(de)$ and $\tau(x) = (v^{-5} + 3v^{-7} + 3v^{-9} + v^{-11})t(ab)t(c)t(de)$.

Although its proof is similar to [**7**, Theorem 5.2.5], the next result is much more general. Its relevance will become clearer in §4.3.

Theorem 3.2.3. *The algebra P_n^A equipped with its canonical basis $\mathbf{B}_n^A$ and the trace τ of Definition 3.2.1 is a tabular algebra with trace.*

Proof. Let Λ be the set of integers r with $0 \leq r \leq n$ and $n - r$ even, ordered in the usual way.

For $\lambda \in \Lambda$, let $(\Gamma(\lambda), B(\lambda))$ be the λ-th tensor power of the table algebra $(A, \mathbf{B})$ with the basis and anti-automorphism induced by Proposition 1.1.5.

Let $M(\lambda)$ be the set of possible configurations of non-propagating edges with endpoints on the line $y = 1$ that arise from an element of $\mathbf{B}_n^A$. (Note that, in this case, the number of non-propagating edges involved will be $(n - \lambda)/2$.) Let $b = b_{i_1} \otimes b_{i_2} \otimes \cdots \otimes b_{i_\lambda}$ be a basis element of $B(\lambda)$ and let m and m' be elements of $M(\lambda)$. The map C produces a basis element in $\mathbf{B}_n^A$ from the triple (m, b, m') as follows. Turn the half-diagram corresponding to m' upside down, reverse the directions of all the arrows and relabel all 1-boxes labelled by $b_i \in \mathbf{B}$ so they are labelled by $b_{\bar{i}}$. Join any free marked points in the line $y = 0$ free marked points in the line $y = 1$ so that they do not intersect. Orient any new edges according to the orientation of the standard n-box. Decorate the λ propagating edges with the basis element b exactly as in the proof of Proposition 2.3.4. (See Example 3.2.4 below for an illustration.)

The map $*$ is as given in Lemma 2.3.2.

It is clear that the image of C as above is the canonical basis of P_n^A. Furthermore, C contains the identity basis element, so axiom (A1) holds. Axiom (A2) follows from Lemma 2.3.2. Axiom (A3) follows from consideration of a product of a basis element a with a basis element $C_{S,T}^b \in \mathbf{c}_\lambda$ in the case where $aC_{S,T}^b$ does not lie in $P_n^A(< \lambda)$. In this case, the structure constants occurring are essentially unaffected by changing the configuration of non-propagating edges at the bottom end of the basis element $C_{S,T}^b$. The part of axiom (A3) guaranteeing independence from $g \in \Gamma(\lambda)$ follows from the associativity of the algebra structure on $(A^{\otimes n}, \mathbf{B}^{\otimes n})$ given by Proposition 2.3.4.

For axiom (A4), we claim that if $D \in \mathbf{c}_\lambda$, we have $\mathbf{a}(D) = a'(D) := (n - \lambda)/2$, i.e., the $\mathbf{a}$-function evaluated at a diagram is half the number of non-propagating edges in that diagram. Let $D = C_{S,T}^b \in \mathbf{c}_\lambda$. By Lemma 2.3.3, $C_{S,S}^1 D = [2]^{a'(D)} C_{S,T}^b$, so $\mathbf{a}(D) \geq a'(D)$. Conversely, the diagram calculus shows that if D' and D'' are canonical basis elements for P_n^A, the number of loops formed in the product $D'D''$ is bounded above both by $a'(D')$ and $a'(D'')$; this implies that the structure constants appearing in $D'D''$ have degree bounded in the same way. Since D can only appear in a product $D'D''$ if $a'(D') \leq a'(D)$ and $a'(D'') \leq a'(D)$, we have $\mathbf{a}(D) \leq a'(D)$. The claim follows.

The above argument also implies that the only way the $\mathbf{a}$-function bound can be achieved is if the three basis elements D', D'', D concerned come from the same $\mathbf{c}_\lambda$. The statement of Lemma 2.3.3 shows that the bound can only be achieved if the pattern of edges at the bottom of D' is the same as the pattern of edges at the top of D'', except that all the directions on the edges have been reversed and all the labels b_i have been changed to $b_{\bar{i}}$. In this case, we may set $D' = C_{S,T}^b$ and $D'' = C_{T,U}^{b'}$, and properties of the diagram calculus give $D'D'' = [2]^{\mathbf{a}(D)} C_{S,U}^{bb'}$. The assertions of axiom (A4) all follow easily.

Finally, we prove axiom (A5). Consider a basis element D. It is clear by symmetry of the definitions that $\tau(D) = \tau(D^*)$, and thus that $\tau(x) = \tau(x^*)$ for all $x \in P_n^A$. To prove the other requirements of the axiom, we note that the diagram corresponding to D has $2k$ non-propagating edges and r propagating edges, where $2k+r = n$ and $k = \mathbf{a}(D)$. To calculate $\tau(D)$, we join each point i to point $2n+1-i$ as described before Definition 3.2.1. An elementary analysis shows that the number of loops formed is at most $k + r = n - \mathbf{a}(D)$, and also that this bound is achieved only if whenever point i is connected to j, we must have point $2n+1-i$ connected to point $2n+1-j$. If there are indeed $k+r$ loops and $\tau(D) \neq 0$, we also require $D = D^*$ (or one of the loops would give trace zero by Lemma 2.3.3) and we require all non-propagating edges to be labelled by 1 (or one of the loops would contain exactly one non-identity box and give trace zero); in other words, $D = C^1_{S,S}$ for some S. If the above bound on loops is achieved and $D = C^1_{S,S}$ for some S, we have $\tau(D) = v^{-n}(v + v^{-1})^{n-\mathbf{a}(D)}$ and it follows that $\tau(v^{\mathbf{a}(D)}D) = 1 \mod v^{-1}\mathcal{A}^-$ as required. In the other cases, the bound is not achieved or $\tau(D) = 0$, and we have $\tau(v^{\mathbf{a}(D)}D) = 0 \mod v^{-1}\mathcal{A}^-$. Axiom (A5) follows, completing the proof. □

Example 3.2.4. Suppose A is a normalized table algebra of rank 3, with $\mathbf{B} = \{1, g, h\}$ and $\bar{g} = h$. (For example, we could take $\mathbf{B} = \mathbb{Z}_3$ and $\bar{\ }$ to be inversion.) Let $n = 8$ and $\lambda = 2$. Let $m, m' \in M(2)$ be as shown in Figures 7 and 8 respectively, and let $b = h \otimes h \in \mathbf{B}^{\otimes 2}$. Then the element $C^b_{m,m'}$ is as shown in Figure 9.

FIGURE 7. The element $m \in M(2)$ of Example 3.2.4

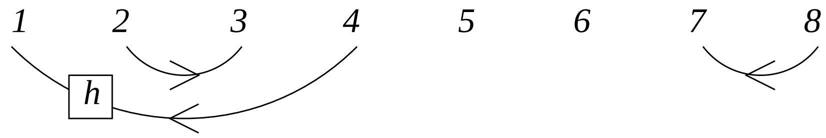


FIGURE 8. The element $m' \in M(2)$ of Example 3.2.4

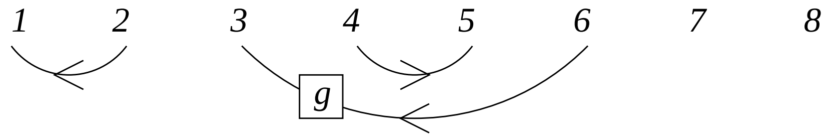
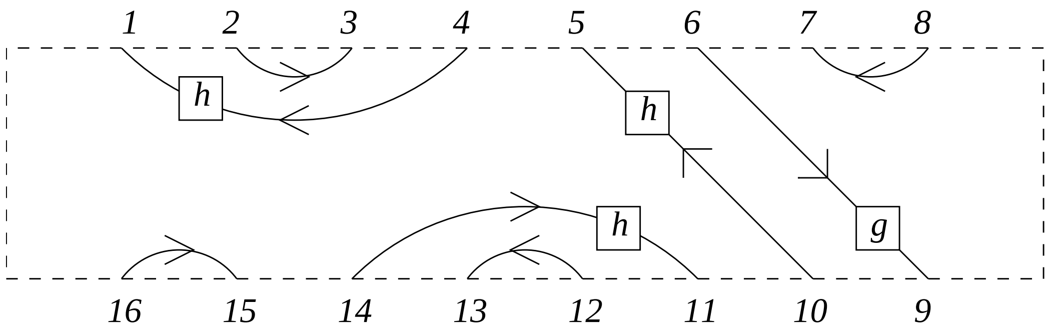


FIGURE 9. The element $C(m, b, m')$ of Example 3.2.4



Remark 3.2.5. The algebra $\mathcal{D}_n^A$ is also a tabular algebra with trace. A suitable table datum may be obtained from the table datum for P_n^A by taking Λ as for P_n^A, considering suitable subsets of $M(\lambda)$, $\Gamma(\lambda)$ and $B(\lambda)$ for each $\lambda \in \Lambda$, and restricting C, $*$ and τ to the appropriate domains. We leave the details to the reader.

4. Relationship with Hecke algebra quotients

In §4.1, we recall the definition of certain Hecke algebra quotients which we call generalized Temperley–Lieb algebras, and the construction of their canonical bases. These bases are related, via the theory of tabular algebras, to the algebras $\mathcal{D}(n, r)$, as we explain in the main results of §4.2.

4.1 Generalized Temperley–Lieb algebras and canonical bases.

Let X be a Coxeter graph, of arbitrary type, and let $W(X)$ be the associated Coxeter group with distinguished set of generating involutions $S(X)$. Denote by $\mathcal{H}(X)$ the Hecke algebra associated to $W(X)$. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ as usual. The $\mathcal{A}$-algebra $\mathcal{H}(X)$ has a basis consisting of elements T_w, with w ranging over $W(X)$, that satisfy

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ qT_{sw} + (q-1)T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

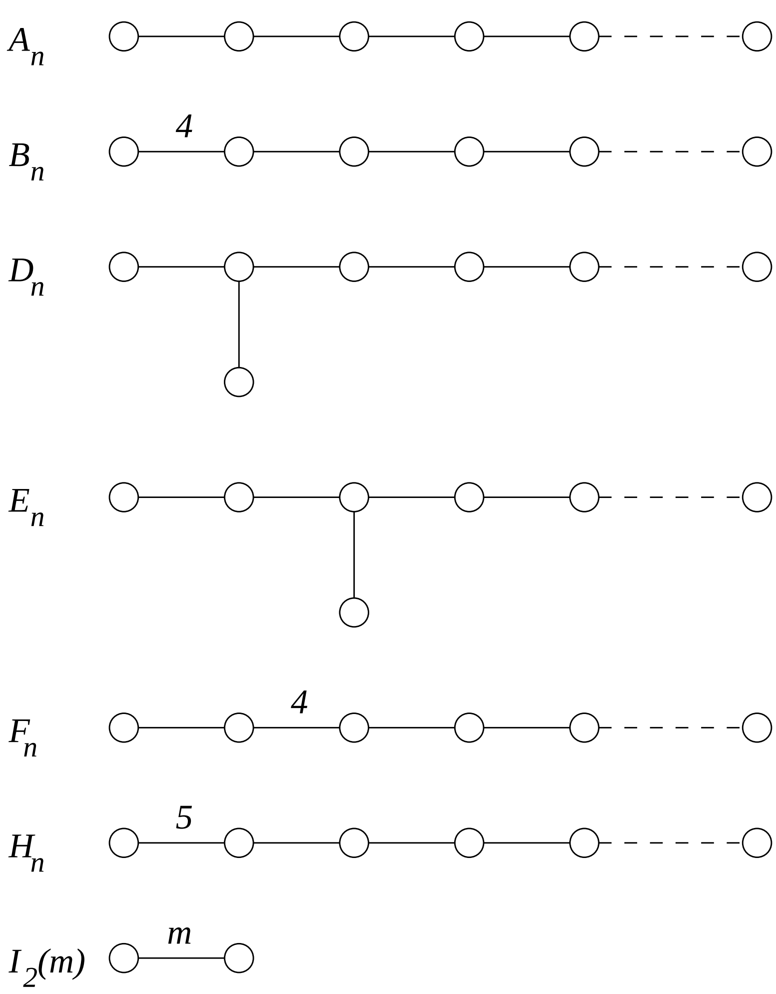
where ℓ is the length function on the Coxeter group $W(X)$, $w \in W(X)$, and $s \in S(X)$. The parameter q is equal to v^2.

Let $J(X)$ be the two-sided ideal of $\mathcal{H}(X)$ generated by the elements

$$\sum_{w \in \langle s, s' \rangle} T_w,$$

where (s, s') runs over all pairs of elements of $S(X)$ that correspond to adjacent nodes in the Coxeter graph, and $\langle s, s' \rangle$ is the group generated by the pair (s, s').

FIGURE 10. Coxeter graphs X
corresponding to finite rank algebras $TL(X)$



(If the nodes corresponding to (s, s') are connected by a bond of infinite strength, then we omit the corresponding relation.)

Definition 4.1.1. Following Graham [**4**, Definition 6.1], we define the generalized Temperley–Lieb algebra $TL(X)$ to be the quotient $\mathcal{A}$-algebra $\mathcal{H}(X)/J(X)$. We denote the corresponding epimorphism of algebras by $\theta : \mathcal{H}(X) \longrightarrow TL(X)$.

The algebra $TL(X)$ may be of finite or infinite rank, and may be of finite rank even when it is the quotient of a Hecke algebra of infinite rank, as the following result shows.

Proposition 4.1.2 (Graham). *The algebra $TL(X)$ corresponding to a Coxeter graph X is of finite rank if and only if its graph is one of those appearing in Figure 10.*

Proof. This is [**4**, Theorem 7.1]. □

Definition 4.1.3. A product $w_1 w_2 \cdots w_n$ of elements $w_i \in W(X)$ is called *reduced* if $\ell(w_1 w_2 \cdots w_n) = \sum_i \ell(w_i)$. We reserve the terminology *reduced expression* for reduced products $w_1 w_2 \cdots w_n$ in which every $w_i \in S(X)$.

Call an element $w \in W(X)$ *complex* if it can be written as a reduced product $x_1 w_{ss'} x_2$, where $x_1, x_2 \in W(X)$ and $w_{ss'}$ is the longest element of some rank 2 parabolic subgroup $\langle s, s' \rangle$ such that s and s' correspond to adjacent nodes in the Coxeter graph.

We define the *content* of $w \in W$ to be the set $c(w)$ of Coxeter generators $s \in S$ that appear in some (any) reduced expression for w. (This can be shown not to depend on the reduced expression chosen, by using the theory of Coxeter groups.)

Denote by $W_c(X)$ the set of all elements of $W(X)$ that are not complex.

Let t_w denote the image of the basis element $T_w \in \mathcal{H}(X)$ in the quotient $TL(X)$.

Proposition 4.1.4 (Graham). *The set $\{t_w : w \in W_c\}$ is an $\mathcal{A}$-basis for the algebra $TL(X)$.*

Proof. See [**4**, Theorem 6.2]. □

We now recall a principal result of [**8**], which establishes the canonical basis for $TL(X)$. This basis is a direct analogue of the important Kazhdan–Lusztig basis of the Hecke algebra $\mathcal{H}(X)$ defined in [**13**].

Fix a Coxeter graph, X. Let $\mathcal{A}^- = \mathbb{Z}[v^{-1}]$, and let $^-$ be the involution on the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ which satisfies $\bar{v} = v^{-1}$.

By [**8**, Lemma 1.4], the algebra $TL(X)$ has a $\mathbb{Z}$-linear automorphism of order 2 that sends v to v^{-1} and t_w to $t_{w^{-1}}^{-1}$. We denote this map also by $^-$.

Let $\mathcal{L}$ be the free $\mathcal{A}^-$-submodule of $TL(X)$ with basis $\{\widetilde{t}_w : w \in W_c\}$, where $\widetilde{t}_w := v^{-\ell(w)} t_w$, and let $\pi : \mathcal{L} \longrightarrow \mathcal{L}/v^{-1}\mathcal{L}$ be the canonical projection.

Proposition 4.1.5. *There exists a unique basis $\{c_w : w \in W_c\}$ for $\mathcal{L}$ such that $\overline{c_w} = c_w$ and $\pi(c_w) = \pi(\widetilde{t}_w)$ for all $w \in W_c$.*

Proof. This is [**8**, Theorem 2.3]. □

The basis $\{c_w : w \in W_c\}$ is called the *canonical basis* (or the *IC basis*) of $TL(X)$. It depends on the t-basis, the involution $^-$, and the lattice $\mathcal{L}$.

In this paper, we shall only be concerned with algebras $TL(X)$ of finite rank. We note that the canonical basis is known in many of these cases: for types A, D and E, see [**8**, Theorem 3.6]; for types B and H, see [**9**, Theorem 2.2.1] and [**6**, Theorem 2.1.3, Theorem 2.2.5]. The type I result follows trivially from [**9**, Proposition 1.2.3].

It is often more convenient to work with an alternate set of algebra generators for $TL(X)$, as follows.

Definition 4.1.6. If $s \in S(X)$, we write $b_s \in TL(X)$ for the element $v^{-1}t_1 + v^{-1}t_s$. It is clear that $TL(X)$ is generated as an algebra by the elements b_s.

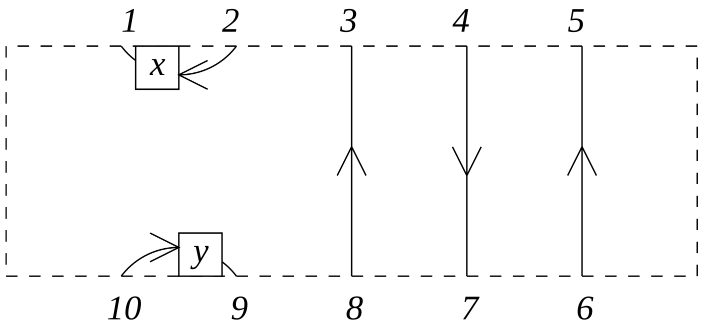
4.2 Main results.

It is convenient at this stage to define some named canonical basis elements of P_n^A that will turn out to be related to the elements b_s of Definition 4.1.6.

Definition 4.2.1. Let A be a normalized table algebra with basis $\mathbf{B}$ and let $n, k \in \mathbb{N}$. Suppose $n > 1$ and $1 \leq k < n$. Let $x \in \mathbf{B}$. Then the canonical basis element $E_k(x)$ of P_n^A is the one where each point i is connected by a vertical edge to point $2n+1-i$, unless $i \in \{k, k+1, 2n-k, 2n+1-k\}$. Points k and $k+1$ are connected by an edge, as are points $2n-k$ and $2n+1-k$. All edges are labelled by $1 \in A$, except the edge connecting k and $k+1$, which is labelled by x, and the edge connecting $2n-k$ and $2n+1-k$, which is labelled by $y = \bar{x}$.

Example 4.2.2. Figure 11 shows the basis element $E_1(x)$ for $n = 5$. We omit the label 1 (see Figure 5) since it occurs frequently.

FIGURE 11. A basis element $E_1(x)$



To describe the sub-tabular algebras that arise from Coxeter systems of various types, we need to define certain subsets of the canonical basis for $P(n, r)$ (see Definition 2.4.5). Our definitions of B-admissible and H-admissible have previously appeared in other forms in [**6**, Definition 2.2.4] and [**5**, Definition 2.2.1] respectively.

Definition 4.2.3. We say an edge in a canonical basis element for $P(n, r)$ is i-*decorated* if it carries a 1-box decorated by the basis element $u_i(x)$ of V_r (see Definition 1.2.2).

- A basis element, of $P(n, 3)$ is said to be B-*admissible* if it is exposed (see Definition 2.4.3) and satisfies one of the following three conditions:

(B1) There is a 0-decorated edge connecting points 1 and $2n$.

(B1′) There is a 2-decorated edge connecting points 1 and $2n$, and at least one non-propagating edge.

(B2) Points 1 and $2n$ are not connected to each other and are each the endpoint of a 1-decorated edge, and there are no other 1-decorated edges.

- A basis element, of $P(n, 4)$ is said to be H-*admissible* if it is exposed and satisfies one of the following two conditions:

(H1) If all edges are propagating, then all edges are 0-decorated.

(H2) If not all edges are propagating, then each edge is either 0-decorated or 2-decorated and both the following statements hold:

(i) either there is a 2-decorated edge connecting nodes 1 and 2 or there is a 0-decorated edge connecting nodes i and $i + 1$ for some $i > 1$;

(ii) either there is a 2-decorated edge connecting nodes $2n$ and $2n - 1$ or there is a 0-decorated edge connecting nodes $2n + 1 - i$ and $2n - i$ for some $i > 1$.

- A basis element, of $P(3, r)$ is said to be I-*admissible* if it is exposed and satisfies one of the following three conditions:

(I1) If all edges are propagating, then all edges are 0-decorated.

(I2) A non-propagating edge is 1-decorated if it is transitional (see Definition 2.2.1) and 0-decorated otherwise.

(I3) A propagating i-decorated edge must have i odd if it is transitional, and i even if it is non-transitional.

Note that there are a total of $2r + 1$ I-admissible basis elements for $P(3, r)$.

Armed with these combinatorial definitions, we can now state the main result.

Theorem 4.2.4.

(i) *Let X be a Coxeter graph of type A_n. There is a monomorphism of $\mathcal{A}$-algebras $\rho_A : TL(A_n) \longrightarrow P(n + 1, 2)$ such that $\rho_A(b_i) = E_i(u_0(x))$. The map ρ_A takes canonical basis elements of $TL(X)$ to canonical basis elements of $P(n + 1, 2)$,* *and its image is the Temperley–Lieb algebra,* $TL(n+1,[2])$.

(ii) *Let* X *be a Coxeter graph of type* B_n. *There is a monomorphism of* $\mathcal{A}$*-algebras* $\rho_B : TL(B_n) \longrightarrow P(n+1,3)$ *such that* $\rho_B(b_1) = E_1(u_1(x))$ *and* $\rho_B(b_i) = E_i(u_0(x))$ *for* $i > 1$. *The map* ρ_B *takes canonical basis elements of* $TL(X)$ *to canonical basis elements of* $P(n+1,3)$, *and its image is the subalgebra of* $\mathcal{D}(n+1,3)$ *spanned by the* B*-admissible basis elements.*

(iii) *Let* X *be a Coxeter graph of type* H_n. *There is a monomorphism of* $\mathcal{A}$*-algebras* $\rho_H : TL(H_n) \longrightarrow P(n+1,4)$ *such that* $\rho_H(b_1) = E_1(u_2(x))$ *and* $\rho_H(b_i) = E_i(u_0(x))$ *for* $i > 1$. *The map* ρ_H *takes canonical basis elements of* $TL(X)$ *to canonical basis elements of* $P(n+1,4)$, *and its image is the subalgebra of* $\mathcal{D}(n+1,4)$ *spanned by the* H*-admissible basis elements.*

(iv) *Let* X *be a Coxeter graph of type* $I_2(m)$ *for* $m > 1$. *There is a monomorphism of* $\mathcal{A}$*-algebras* $\rho_I : TL(I_2(m)) \longrightarrow P(3,m-1)$ *such that* $\rho_I(b_1) = E_1(u_1(x))$ *and* $\rho_I(b_2) = E_2(u_0(x))$. *The map* ρ_I *takes canonical basis elements of* $TL(X)$ *to canonical basis elements of* $P(3,m-1)$, *and its image is the subalgebra of* $\mathcal{D}(3,m-1)$ *spanned by the* I*-admissible basis elements.*

Proof. We prove each part separately.

For (i), we note that the existence of ρ_A and the description of its image are well known; see [**11**, Example 2.1]. The canonical basis of $TL(A_n)$ is shown in [**8**, Theorem 3.6] to coincide with the so-called "monomial basis" which in turn coincides with the basis of diagrams of Definition 2.1.5 by [**3**, Proposition 3.2.2]. (The diagrams in [**3**] are not oriented, but the difference is cosmetic in this case.)

For (ii), we maintain the notation of Lemma 1.2.5 and observe using a case by case check that the B-admissible basis elements span a subalgebra, A', of $P(n+1,3)$ (and hence of $\mathcal{D}(n+1,3)$, since all B-admissible basis elements are exposed). Another case by case check shows that when relation (d) of Definition 2.2.2 is used when calculating the product of two B-admissible basis elements, the label on the loop (see Figure 6) is always a linear combination of 1 and z. By Lemma 1.2.5, there is a map, ψ, from A' to $\mathcal{D}(n+1,2)$ obtained by replacing each box decorated by $x \in V_3$ by a box decorated by $\psi(x) \in V_2$. Since $t(\phi(1)) = 1 = t(1')$ and $t(\phi(z)) = 0 = t(z')$, where t is the trace of Proposition 1.1.4, it follows from Definition 2.2.2 that ψ is a homomorphism. From Definition 4.2.3, we see that the labels on a B-admissible basis element may be reconstructed from a knowledge of which edges are not 0-decorated. It follows that ψ sends the canonical basis of A' to a linearly independent set in $\mathcal{D}(n+1,2)$, and hence ψ is injective.

The tangles of [**6**, §2.2] may be easily reconstructed from ψ by ignoring all the arrows (which convey no information as the involution on V_r is the identity map), replacing z' by a square decoration and replacing $\frac{1'+z'}{2} = \frac{\psi(y)}{\sqrt{2}}$ by a round decoration. The relations in figures 4 and 5 of [**6**, §2.2] then correspond to the identities:

$$\begin{aligned}
\delta t(1) &= \delta, \\
\delta t(\psi(y/\sqrt{2})) &= \frac{\delta}{2}, \\
\psi((y/\sqrt{2})^2) &= \psi(y/\sqrt{2}), \\
\psi(z) &= 2\psi(y/\sqrt{2}) - \psi(1), \\
t(\psi(z)) &= 0, \\
\psi(z)^2 &= \psi(1), \\
\psi(y/\sqrt{2})\psi(z) &= \psi(y/\sqrt{2}).
\end{aligned}$$

Using these identifications, we find that the statement of part (ii) is a restatement of [**6**, Theorem 2.2.5] and the map ρ_B agrees with that of [**6**, Theorem 2.2.3].

The proof of (iii) is similar to, but much easier than, (ii). The result is a restatement of [**6**, Theorem 2.1.2], where the arrows are ignored as before and a decorated edge in [**6**, §2.1] is identified with a 2-decorated edge. The relations, shown in [**5**, Figure 5], correspond to the relations

$$\begin{aligned}
\delta t(u_0(x)) &= \delta, \\
t(u_2(x)) &= 0, \\
u_2(x)^2 &= u_0(x) + u_2(x).
\end{aligned}$$

in V_4.

To prove (iv), it is convenient to construct an explicit bijection from the set of I-admissible diagrams to W_c. In this case, $S(I_2(m)) = \{s_1, s_2\}$. We send the element described in condition (I1) of Definition 4.2.3 to $1 \in W_c$. Let D be a nonidentity I-admissible diagram. Then D has a unique propagating edge, which is i-decorated for some i, and D corresponds to the unique element $w \in W_c$ satisfying:

(a) $s_1 w < w$ if and only if points 1 and 2 are connected by an edge;

(b) $s_2 w < w$ if and only if points 2 and 3 are connected by an edge;

(a′) $ws_1 < w$ if and only if points 5 and 6 are connected by an edge;

(b′) $ws_2 < w$ if and only if points 4 and 5 are connected by an edge;

(c) $\ell(w) = i + 1$.

Let $s \in S(I_2(m))$ with $\{s, s'\} = \{s_1, s_2\}$. Using standard properties of Hecke algebras, we find that $c_s c_w = [2]c_w$ if $sw < w$. Suppose now that $sw > w$. If $\ell(w) \notin \{m - 1, 0, 1\}$, we have

$$c_s c_w = c_{sw} + c_{s'w}.$$

If $\ell(w) = 1$ then $w = s'$ and $c_s c_{s'} = c_{ss'}$. If $\ell(w) = m - 1$ then $sw = w_0$ and $c_s c_w = c_{s'w}$. Using the recurrence relation in Definition 1.2.1, we see that the correspondence of the previous paragraph defines an isomorphism of $\mathcal{A}$-algebras sending canonical basis elements to I-admissible diagrams. □

In fact, it is possible to state a result similar to Theorem 4.2.4 more concisely, as follows.

Theorem 4.2.5. *Let X be a (connected) Coxeter graph of rank $n > 1$ of type A, B, H or I, and let $m > 2$ be the highest bond label. Then there is a monomorphism of $\mathcal{A}$-algebras $\rho : TL(X) \longrightarrow P(n + 1, m - 1)$ such that $\rho(b_1) = E_1(u_1(x))$ and $\rho(b_i) = E_i(u_0(x))$ for $i > 1$. The map ρ takes canonical basis elements of $TL(X)$ to canonical basis elements of $\mathcal{D}(n + 1, m - 1)$, and thus $TL(X)$ equipped with its canonical basis is a sub-tabular algebra.*

Proof. The assertion about sub-tabular algebras is a consequence of Theorem 3.2.3. The other assertions, in the case of type B and type I, are dealt with by Theorem 4.2.4 (ii) and (iv) respectively.

In type A, we take $\rho := \omega \circ \rho_A$, and in type H, we take $\rho := \omega \circ \rho_H$. By Lemma 2.4.6, ω is an isomorphism permuting the canonical basis, so ρ is a monomorphism taking canonical basis elements to canonical basis elements by Theorem 4.2.4. It remains to check that ρ has the correct effect on the generators b_i (in particular, on b_1), but this follows easily from the definitions of ω, ρ_A and ρ_H. □

Remark 4.2.6. Although the statement of Theorem 4.2.5 is much shorter than that of Theorem 4.2.4, there are situations where the results of Theorem 4.2.4 are more useful. This is because the algebras $TL(X)$ where X is of type A or H are tabular algebras (see [**7**, §§4, 5]), and the table datum may be obtained easily from that of $\mathcal{D}(n,r)$ by appropriate restriction of the identifications in Theorem 4.2.4.

As well as being more uniform, the statement of Theorem 4.2.5 gives well-defined maps for the Coxeter systems $A_2 = I_2(3)$, $B_2 = I_2(4)$ and $H_2 = I_2(5)$.

It may be easily checked that $\omega \circ \rho = \rho$ for Coxeter systems of type B or type $I_2(m)$ with m even. For $X = I_2(m)$ with m odd, consideration of $\omega \circ \rho$ produces a tabular structure for $TL(X)$ analogous to that obtained from Theorem 4.2.4 in types A and H. The details are not hard to fill in.

4.3 Applications.

One of the motivations for Theorem 3.2.3 relates to the following result.

Theorem 4.3.1 [**7**]**.** *Let A be a tabular algebra (over $\mathcal{A}$) with trace τ and table datum $(\Lambda, \Gamma, B, M, C, *)$. Then the map $(x, y) \longrightarrow \tau(xy^*)$ defines a symmetric, nondegenerate bilinear form on A with the following properties.*

(i) *For all $x, y, z \in A$, $(x, yz) = (xz^*, y)$.*

(ii) *The tabular basis is almost orthonormal with respect to this bilinear form: whenever $X, X' \in \mathrm{Im}(C)$, we have*

$$(X, X') = \begin{cases} 1 \mod v^{-1}\mathcal{A}^- & \text{if } X = X', \\ 0 \mod v^{-1}\mathcal{A}^- & \text{otherwise.} \end{cases}$$

Proof. This is [**7**, Theorem 2.2.5]. □

These bilinear forms have direct relevance to generalized Temperley–Lieb algebras, where the map $*$ is defined as follows.

Lemma 4.3.2. Let X be an arbitrary Coxeter system. There is a unique $\mathcal{A}$-linear anti-automorphism, $*$, of $TL(X)$ that fixes the generators $\{b_s : s \in S(X)\}$ of Definition 4.1.6.

If X is of type A, B, H or I, the map $*$ is induced by the tabular anti-automorphism $*$ of $P(n,r)$ via the monomorphism ρ of Theorem 4.2.5.

Proof. It is well-known that there is an $\mathcal{A}$-linear automorphism of $\mathcal{H}(X)$ sending C'_s to itself for any $s \in S(X)$. (This is the $\mathcal{A}$-linear anti-automorphism that sends T_w to $T_{w^{-1}}$.) This map fixes the generators of the ideal $J(X)$ and thus gives an automorphism of $TL(X)$ fixing $b_s = \theta(C'_s)$, where θ is as in Definition 4.1.1. Uniqueness follows as $\{b_s : s \in S(X)\}$ is a set of algebra generators for $TL(X)$.

For the second assertion, we note that $\rho(b_i)^* = E_i(u_{\delta_{i0}})^* = E_i(u_{\delta_{i0}}) = \rho(b_i)$. $\square$

The next result is a generalization of the previously unproven [**6**, Hypothesis 5.3.1].

Corollary 4.3.3. *Let X be a Coxeter system of type A, B, H or I, and let $*$ denote the anti-automorphism of $TL(X)$ given in Lemma 4.3.2. Then there is a a symmetric, nondegenerate bilinear form, $(,)$, on $TL(X)$ with the following properties.*

(i) *For all $x, y, z \in TL(X)$, $(x, yz) = (xz^*, y)$.*

(ii) *The canonical basis of $TL(X)$ is almost orthonormal with respect to this bilinear form: whenever $w, w' \in W_c(X)$ and c_w and $c_{w'}$ are the corresponding canonical basis elements, we have*

$$(c_w, c_{w'}) = \begin{cases} 1 \mod v^{-1}\mathcal{A}^- & \text{if } w = w', \\ 0 \mod v^{-1}\mathcal{A}^- & \text{otherwise.} \end{cases}$$

(iii) *The basis $\{\widetilde{t}_w : w \in W_c\}$ of $TL(X)$ defined in §4.1 is almost orthonormal with respect to this bilinear form: whenever $w, w' \in W_c(X)$, we have*

$$(\widetilde{t}_w, \widetilde{t}_{w'}) = \begin{cases} 1 \mod v^{-1}\mathcal{A}^- & \text{if } w = w', \\ 0 \mod v^{-1}\mathcal{A}^- & \text{otherwise.} \end{cases}$$

Proof. Theorem 4.2.5 shows that the algebra $TL(X)$ is a sub-tabular algebra of a tabular algebra with trace, with respect to the involution $*$ and its canonical basis. The bilinear form $(,)$ is therefore inherited from the bilinear form on $P(n, r)$ arising from theorems 3.2.3 and 4.3.1. Theorem 4.3.1 also shows that tabular algebras have the properties referred to in parts (i) and (ii). The truth of (i) and (ii) is now evident from the definition of sub-tabular algebras.

Part (iii) follows from (ii) and Proposition 4.1.5, which shows that $\widetilde{t}_w$ and c_w agree modulo $v^{-1}\mathcal{L}$. $\square$

Corollary 4.3.3 leads to the following characterization of the canonical basis, up to sign.

Proposition 4.3.4. *Let X be a Coxeter system of type A, B, H or I, and let $TL(X)$ be the corresponding generalized Temperley–Lieb algebra over $\mathcal{A}$ equipped* *with the bilinear form of Corollary 4.3.3 and the automorphism* $^-$ *of* §4.1. *Suppose* $x \in TL(X)$ *is such that* $\bar{x} = x$ *and* $(x, x) = 1 \mod v^{-1}\mathcal{A}^-$. *Then either* x *or* $-x$ *is a canonical basis element* c_w.

Proof. Write $x = \sum_{w \in W_c} \lambda_w c_w$, where $\lambda_w \in \mathcal{A}$. Since $\bar{x} = x$ and $\overline{c_w} = c_w$ by Proposition 4.1.5, we must have $\overline{\lambda_w} = \lambda_w$ for all w, i.e., the coefficient of v^k in λ_w is the same as the coefficient of v^{-k}. Define $j = j(x)$ to be $\max_{w \in W_c, \lambda_w \neq 0} \deg(\lambda_w)$; this is well-defined because $x \neq 0$ by the hypothesis $(x, x) = 1$.

Now $v^{-j}x = \sum_{w \in W_c} \lambda'_w c_w$, where $\lambda'_w \in \mathcal{A}^-$ for all w. Define $\mu_w \in \mathbb{Z}$ to be the constant coefficient of λ'_w. Let

$$Z := \{w \in W_c : \mu_w \neq 0\} = \{w \in W_c : \lambda'_w \notin v^{-1}\mathcal{A}^-\}.$$

The choice of j guarantees that Z is nonempty. Since $(c_w, c_{w'}) = 0 \mod v^{-1}\mathcal{A}^-$ if $w \neq w'$, we have

$$(v^{-j}x, v^{-j}x) \equiv \left(\sum_{w \in Z} (\lambda'_w)^2 (c_w, c_w) \right) + \left(\sum_{w' \in W_c \setminus Z} (\lambda'_{w'})^2 (c_{w'}, c_{w'}) \right) \mod v^{-1}\mathcal{A}^-.$$

Applying Corollary 4.3.3 (ii), we see that the second sum lies in $v^{-1}\mathcal{A}^-$ and that

$$\sum_{w \in Z} (\lambda'_w)^2 (c_w, c_w) \equiv \sum_{w \in Z} (\mu_w)^2 \mod v^{-1}\mathcal{A}^{-1}.$$

By hypothesis, $(x, x) = 1 \mod v^{-1}\mathcal{A}$, which forces $j = 0$, $|Z| = 1$ and $\mu_w = \pm 1$ for $w \in Z$. We conclude that x is a canonical basis element if $\mu_w = 1$, and $-x$ is a canonical basis element if $\mu_w = -1$. $\square$

Remark 4.3.5. Results such as Proposition 4.3.4, where a canonical basis is characterized up to sign by an almost orthonormality property for a natural inner product are familiar from the work of Kashiwara [**12**] and Lusztig [**15**, Theorem 14.2.3] on quantized enveloping algebras.

A result similar to Proposition 4.3.4 is true for the Kazhdan–Lusztig basis $\{C'_w : w \in W\}$ [**13**] for the Hecke algebra, although Corollary 4.3.3 and Proposition 4.3.4 are not obvious consequences of this result. We offer the following conjecture concerning the role of the Kazhdan–Lusztig basis in this context.

Conjecture 4.3.6. *Let* X *be a (connected) Coxeter graph of rank* $n > 1$ *of type* A, B, H *or* I, *and let* $m > 2$ *be the highest bond label. Then there is a homomorphism* *of $\mathcal{A}$-algebras $\rho : \mathcal{H}(X) \longrightarrow P(n+1, m-1)$ such that $\rho(C'_{s_1}) = E_1(u_1(x))$ and $\rho(C'_{s_i}) = E_i(u_0(x))$ for $i > 1$. The map ρ is injective on the set*

$$\{C'_w : w \in W(X),\ \rho(C'_w) \neq 0\}$$

and the image of this set under ρ is a set of canonical basis elements of $\mathcal{D}(n+1, m-1)$.

The results of [**10**, §3.1] in conjunction with Theorem 4.2.5 show that Conjecture 4.3.6 is true when the Coxeter group is finite, so the only open cases are Coxeter systems of type H_n for $n > 4$.

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