ACYCLIC HEAPS OF PIECES, II

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Abstract. We characterize and classify the "regular classes of heaps" introduced by the author using ideas of Fan and of Stembridge. The irreducible objects fall into five infinite families with one exceptional case.

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Introduction

A heap is an isomorphism class of labelled posets satisfying certain axioms. Heaps have a wide variety of applications, as discussed by Viennot in [**13**].

In [**9**], the author studied combinatorial properties (called properties P1 and P2) which may or may not hold for a given heap; property P1 is based on Fan's (algebraic) notion of cancellability [**3**, §4] and property P2 is related to Stembridge's definition of full commutativity [**12**, §1]. When certain results of Fan from [**3**] and [**4**, §3] are reinterpreted in the context of heaps, we find that in certain heap monoids

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$H(P, \mathcal{C})$ property P2 implies property P1. In this case, we call $H(P, \mathcal{C})$ a "regular class of heaps". The basic combinatorial and linear properties of regular classes of heaps were developed in [**9**, §2].

The combinatorial properties of regular classes of heaps were used by Fan [**3**, §6] to investigate the representation theory of certain Hecke algebra quotients (also known as generalized Temperley–Lieb algebras). Although slightly more subtle, the linear properties of regular classes of heaps are closely related to the combinatorial properties. This theory was first developed in many important cases by Graham in his thesis [**5**] using a direct combinatorial argument, but our approach seems to make the proofs more transparent as well as more general. Graham used these properties to obtain results on structure constants for the Kazhdan–Lusztig bases of certain Hecke algebras [**5**, §9]. In [**10**], J. Losonczy and the author used the same properties to prove that in certain cases, the Kazhdan–Lusztig type bases of certain generalized Temperley–Lieb algebras are given by monomials in the generators. The theory can also be applied to certain diagram calculi for these algebras. For further applications and more details, the reader is referred to [**9**, §4.1].

In the light of these properties, it is desirable to obtain a better understanding of regular classes of heaps. The first main result of this paper (Theorem 1.5.1, proved in §2) gives some equivalent characterizations of the property of being regular; these involve linear algebra and certain associative algebras as well as combinatorial properties. The second main result (Theorem 1.5.2, proved in §3) solves the problem posed in [**9**, Problem 4.3.1] and gives a complete classification of regular classes of heaps, assuming the corresponding set of pieces is finite. These are classified by their concurrency graphs, and the irreducible objects fall into five infinite families together with one exceptional case.

1. Preliminaries

We begin with some preliminary material that is necessary for the statement of the main results. Our approach follows [**13**] and [**9**].

1.1 Heaps.

We start by recalling the basic definitions following the conventions of [**9**]. These differ slightly from those of [**13**]; see [**9**, §1.1] for details.

Definition 1.1.1. Let P be a set equipped with a symmetric and reflexive binary relation $\mathcal{C}$. The elements of P are called *pieces*, and the relation $\mathcal{C}$ is called the *concurrency relation.*

A *labelled heap* with pieces in P is a triple $(E, \leq, \varepsilon)$ where $(E, \leq)$ is a finite (possibly empty) partially ordered set with order relation denoted by $\leq$ and ε is a map $\varepsilon : E \longrightarrow P$ satisfying the following two axioms.

1. For every $a, b \in E$ such that $\varepsilon(a) \ \mathcal{C} \ \varepsilon(b)$, a and b are comparable in the order $\leq$.
2. The order relation $\leq$ is the transitive closure of the relation $\leq_{\mathcal{C}}$ such that for all $a, b \in E$, $a \ \leq_{\mathcal{C}} \ b$ if and only if both $a \leq b$ and $\varepsilon(a) \ \mathcal{C} \ \varepsilon(b)$.

The terms *minimal* and *maximal* applied to the elements of the labelled heap refer to minimality (respectively, maximality) with respect to $\leq$.

Example 1.1.2. Let $P = \{1, 2, 3\}$ and, for $x, y \in P$, define $a \ \mathcal{C} \ b$ if and only if $|x - y| \leq 1$. Let $E = \{a, b, c, d, e\}$ partially ordered by extension of the (covering) relations $a \leq c$, $b \leq c$, $c \leq d$, $c \leq e$. Define the map ε by the conditions $\varepsilon(a) = \varepsilon(d) = 1$, $\varepsilon(c) = 2$ and $\varepsilon(b) = \varepsilon(e) = 3$. Then $(E, \leq, \varepsilon)$ can easily be checked to satisfy the axioms of Definition 1.1.1 and it is a labelled heap. The minimal elements are a and b, and the maximal elements are d and e.

Definition 1.1.3. Let $(E, \leq, \varepsilon)$ and $(E', \leq', \varepsilon')$ be two labelled heaps with pieces in P and with the same concurrency relation, $\mathcal{C}$. An isomorphism $\phi : E \longrightarrow E'$ of posets is said to be an *isomorphism of labelled posets* if $\varepsilon = \varepsilon' \circ \phi$.

A *heap* of pieces in P with concurrency relation $\mathcal{C}$ is a labelled heap (Definition 1.1.1) defined up to labelled poset isomorphism. The set of such heaps is denoted by $H(P, \mathcal{C})$. We denote the heap corresponding to the labelled heap $(E, \leq, \varepsilon)$ by $[E, \leq, \varepsilon]$.

We will sometimes abuse language and speak of the underlying set of a heap, when what is meant is the underlying set of one of its representatives.

Definition 1.1.4. Let $(E, \leq, \varepsilon)$ be a labelled heap with pieces in P and F a subset of E. Let ε' be the restriction of ε to F. Let $\mathcal{R}$ be the relation defined on F by $a \mathcal{R} b$ if and only if $a \leq b$ and $\varepsilon(a) \mathcal{C} \varepsilon(b)$. Let $\leq_F$ be the transitive closure of $\mathcal{R}$. Then $(F, \leq_F, \varepsilon')$ is a labelled heap with pieces in P. The heap $[F, \leq_F, \varepsilon']$ is called a *subheap* of $[E, \leq, \varepsilon]$. If $v \in E$, we let $E(v) = [E(v), \leq_{E(v)}, \varepsilon']$ be the subheap of E obtained by defining $E(v) = E \backslash \{v\}$.

We will often implicitly use the fact that a subheap is determined by its set of vertices and the heap it comes from.

Definition 1.1.5. The *concurrency graph* associated to the class of heaps $H(P, \mathcal{C})$ is the graph whose vertices are the elements of P and for which there is an edge from $v \in P$ to $w \in P$ if and only if $v \neq w$ and $v \mathcal{C} w$.

Definition 1.1.6. Let $E = [E, \leq_E, \varepsilon]$ and $F = [F, \leq_F, \varepsilon']$ be two heaps in $H(P, \mathcal{C})$. We define the heap $G = [G, \leq_G, \varepsilon''] = E \circ F$ of $H(P, \mathcal{C})$ as follows.

1. The underlying set G is the disjoint union of E and F.
2. The labelling map ε'' is the unique map $\varepsilon'' : G \longrightarrow P$ whose restriction to E (respectively, F) is ε (respectively, ε').
3. The order relation $\leq_G$ is the transitive closure of the relation $\mathcal{R}$ on G, where $a \mathcal{R} b$ if and only if one of the following three conditions holds:

(i) $a, b \in E$ and $a \leq_E b$;
(ii) $a, b \in F$ and $a \leq_F b$;
(iii) $a \in E,\ b \in F$ and $\varepsilon(a) \mathcal{C} \varepsilon'(b)$.

Remark 1.1.7. Definition 1.1.6 can easily be shown to be sound (see [**13**, §2]). It is immediate from the construction that E and F are subheaps of $E \circ F$.

As in [**13**], we will write $a \circ E$ and $E \circ a$ for $\{a\} \circ E$ and $E \circ \{a\}$, respectively. Note that $a \circ E$ and $b \circ E$ are equal as heaps if $\varepsilon(a) = \varepsilon(b)$. If I is a finite set, we may also write $\prod_{i \in I} a_i$ for the product of the heaps $\{a_i\}$ in the case where the singleton heaps commute pairwise.

Definition 1.1.8. A *trivial heap* is a heap $[E, \leq, \varepsilon]$ for which the order relation $\leq$ is trivial.

1.2 Convex chains and regular classes.

In §1.2, we recall the definitions of properties P1 and P2 for heaps from [**9**, §2.2], and we recall some terminology relating to convex chains from [**9**, §2.3].

Definition 1.2.1. A *convex chain* in a heap $E = [E, \leq, \varepsilon]$ is a chain

$$\mathbf{c} = (x_1, x_2, \dots, x_t) : x_1 < x_2 < \cdots < x_t$$

of vertices in E such that whenever $x_i < y < x_j$ for some y, the vertex y is an element of the chain. A convex chain is said to be *balanced* if $\varepsilon(x_1) = \varepsilon(x_t)$. If $\mathbf{c}$ is a balanced convex chain, we define the heap $E/\mathbf{c}$ to be the subheap of E obtained by omitting the vertices $x_2, x_3, \dots x_t$. We call the heap $E/\mathbf{c}$ the *contraction of* E *along* $\mathbf{c}$, and the number t is called the *length* of the chain.

We can improve on the notation of [**9**] by using the following notation (based on [**3**, §2]) for maximal and minimal elements of a heap.

Definition 1.2.2. Let $E = [E, \leq, \varepsilon]$ be a heap. We define $\mathcal{L}(E)$ to be the set of vertices minimal in E, and $\mathcal{R}(E)$ to be the set of vertices maximal in E.

Example 1.2.3. If E is the heap arising from Example 1.1.2, we have $\mathcal{L}(E) = \{a, b\}$ and $\mathcal{R}(E) = \{c, d\}$.

Definition 1.2.4 (Property P1). Let $E = [E, \leq, \varepsilon] \in H(P, \mathcal{C})$ be a heap. We write $E(a) \prec^+ E$ (respectively, $E(a) \prec^- E$) if $a \in \mathcal{R}(E)$ (respectively, $a \in \mathcal{L}(E)$)

and there exists $b \in \mathcal{R}(E(a))\backslash\mathcal{R}(E)$ (respectively, $b \in \mathcal{L}(E(a))\backslash\mathcal{L}(E)$) with $\varepsilon(b) \neq \varepsilon(a)$. We write $E(a) \prec E$ if either $E(a) \prec^+ E$ or $E(a) \prec^- E$.

If there is a (possibly trivial) sequence $E_1 \prec E_2 \prec \cdots \prec E$ of heaps in $H(P, \mathcal{C})$ where E_1 is a trivial heap, we say that the heap E is *dismantlable* or that E has property P1.

Remark 1.2.5. Property P1 is closely related to Fan's notion of left and right cancellability [**3**, Definition 4.2.4]. As in [**9**], we avoid the term "cancellability" in this paper because of possible confusion with the use of this term in the theory of monoids.

Definition 1.2.6 (Property P2). We say a heap $E = [E, \leq, \varepsilon] \in H(P, \mathcal{C})$ has property P2 if it contains no balanced convex chains of length 2 or 3.

Remark 1.2.7. Property P2 is modelled on Stembridge's characterization of full commutativity [**12**, Proposition 2.3] in the case of a simply laced Coxeter group.

1.3 Acyclic heaps.

We recall the definition of the map ∂ from [**9**, §1.2], to which the reader is referred for further elaboration and examples. Throughout §1.3, we let $[E, \leq, \varepsilon]$ be a heap in the set $H(P, \mathcal{C})$ with pieces in P and concurrency relation $\mathcal{C}$. We also fix a field, k.

Definition 1.3.1. Let V_0 be the set of elements of $[E, \leq, \varepsilon]$, i.e., the set of elements of (a representative of) the underlying poset, E. We call the elements of V_0 *vertices* and denote their k-span by C_0.

Let V_1 be the set of all pairs $(x, y) \in E \times E$ with $x < y$ and $\varepsilon(x) = \varepsilon(y)$ such that there is no element z for which we have both $\varepsilon(x) = \varepsilon(z) = \varepsilon(y)$ and $x < z < y$. We call the elements of V_1 *edges* and denote their k-span by C_1.

The k-linear map $\partial = \partial_E : C_1 \longrightarrow C_0$ is defined by its effect on the edges as follows:

$$\partial : (x, y) \mapsto \sum_{\substack{x < w < y \\ \varepsilon(w) \ \mathcal{C} \ \varepsilon(x)}} w.$$

Definition 1.3.2. Let $E = [E, \leq_E, \varepsilon]$ be a heap in $H(P, \mathcal{C})$ and let k be a field. We say E is *acyclic* if $\ker \partial_E = 0$. We say E is *strongly acyclic* if E is acyclic and $E(v)$ is acyclic for all $v \in E$.

Some of the main results of [**9**] may be summarized in the following theorem.

Theorem 1.3.3 [**9**]**.**

(a) *Any strongly ayclic heap has property P2.*

(b) *Any heap with property P1 is acyclic.*

(c) *Let $H(P, \mathcal{C})$ be a regular class of heaps, and let $E = [E, \leq, \varepsilon]$ be a heap of $H(P, \mathcal{C})$. Then:*

(i) *E has property P2 if and only if it is strongly acyclic;*

(ii) *E has property P1 if and only if it is acyclic.*

Proof. For (a), see [**9**, Proposition 2.2.7]. For (b), see [**9**, Proposition 2.2.3]. For (c), see [**9**, Theorem 2.4.2] for (i) and [**9**, Theorem 2.4.4] for (ii). □

Remark 1.3.4. Our characterization of regular classes of heaps in Theorem 1.5.1 will show that the criterion in Theorem 1.3.3 (c)(i) is equivalent to $H(P, \mathcal{C})$ being regular. The same is not true of Theorem 1.3.3 (c)(ii), although we do not prove this here.

1.4 Algebras arising from heaps.

It is sometimes convenient to phrase arguments about the combinatorics or linear properties of heaps in an algebraic way. For this, we recall the definition of certain monoids and algebras discussed in [**9**, §3.1].

Definition 1.4.1. A class of heaps $H(P, \mathcal{C})$ has a natural monoid structure with composition given by the map $\circ$ of Definition 1.1.6. We call this monoid the *heap monoid*.

Another way to approach the heap monoid is by considering the commutation monoids of Cartier and Foata [**1**], which are defined as follows.

Definition 1.4.2. Let A be a set and let A^* be the free monoid generated by A. Let C be a symmetric and antireflexive relation on A. The *commutation monoid* $\mathrm{Co}(A, C)$ is the quotient of the free monoid A^* by the congruence $\equiv_C$ generated by the commutation relations:

$$ab \equiv_C ba \text{ for all } a, b \in A \text{ with } a \; C \; b.$$

The following result, proved in [**13**, Proposition 3.4], shows that the heap monoid is naturally isomorphic to a commutation monoid.

Proposition 1.4.3. *Let $H(P, \mathcal{C})$ be a class of heaps and let C be the complementary relation of $\mathcal{C}$. Let $E = [E, \leq, \varepsilon]$ be a heap of $H(P, \mathcal{C})$. The map from $H(P, \mathcal{C})$ to $\mathrm{Co}(P, C)$ that sends the heap*

$$a_1 \circ a_2 \circ \cdots \circ a_r \mapsto \varepsilon(a_1)\varepsilon(a_2)\cdots\varepsilon(a_r)$$

is an isomorphism of monoids. $\square$

It is convenient to study certain quotients of heap monoid algebras. The algebras below and their bases appear in the work of Fan [**2**] and Graham [**5**].

Definition 1.4.4. Maintain the above notation, so that C is the complementary relation of $\mathcal{C}$. Let $\mathcal{A}$ be the ring of Laurent polynomials $\mathbb{Z}[v, v^{-1}]$, let $\delta := v + v^{-1}$, and let $\mathcal{A}\mathrm{Co}(P, C)$ be the monoid algebra of $\mathrm{Co}(P, C)$ over $\mathcal{A}$. We define the *generalized Temperley–Lieb algebra* $TL(P, \mathcal{C})$ to be the $\mathcal{A}$-algebra obtained by quotienting $\mathcal{A}\mathrm{Co}(P, C)$ by the relations

$$ss = \delta s,$$

$$sts = s \text{ if } s \neq t \text{ and } s \; \mathcal{C} \; t,$$

where $s, t \in P$.

Lemma 1.4.5. *The isomorphism of Proposition 1.4.3 induces an isomorphism between the algebra $TL(P, \mathcal{C})$ and the quotient of $\mathcal{A}H(P, \mathcal{C})$ by the relations*

$$E = \delta E/\mathbf{c} \text{ if } \mathbf{c} \text{ is a balanced convex chain of length 2,}$$

$$E = E/\mathbf{c} \text{ if } \mathbf{c} \text{ is a balanced convex chain } x < y < z \text{ with } \varepsilon(x) \neq \varepsilon(y).$$

Proof. See [**9**, Lemma 3.1.6]. □

Proposition 1.4.6. *The quotient of $\mathcal{A}H(P, \mathcal{C})$ described in Lemma 1.4.5 has as a free $\mathcal{A}$-basis the images of those heaps in $H(P, \mathcal{C})$ with property P2.*

Proof. See [**9**, Proposition 3.2.2]. □

Definition 1.4.7. The basis of $TL(P, \mathcal{C})$ corresponding to the basis of $\mathcal{A}H(P, \mathcal{C})$ given in Proposition 1.4.6 under the isomorphism of Proposition 1.4.3 is called the *monomial basis* of $TL(P, \mathcal{C})$. (It consists of certain monomials in the set P.)

1.5 Statement of main results.

The main two results of this paper concern the characterization and classification of regular classes of heaps. §2 will be devoted to Theorem 1.5.1, and §3 will be devoted to Theorem 1.5.2.

Theorem 1.5.1. *Let $H(P, \mathcal{C})$ be the class of heaps with pieces in P and concurrency relation $\mathcal{C}$. The following are equivalent:*

(i) *every heap in $H(P, \mathcal{C})$ with property P2 has property P1 (i.e., $H(P, \mathcal{C})$ is regular);*

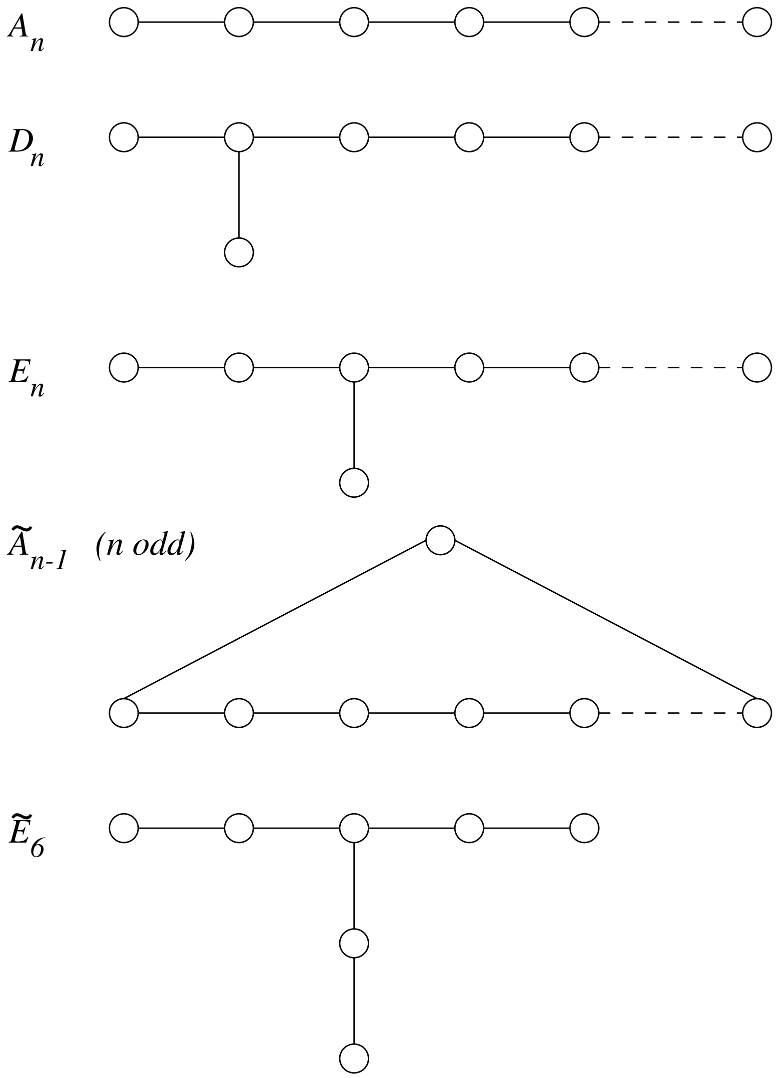
(ii) *every heap in $H(P, \mathcal{C})$ with property P2 is strongly acyclic;*

(ii′) *property P2 and the property of being strongly acyclic are equivalent for heaps in $H(P, \mathcal{C})$;*

(iii) *if E is a heap with property P2 then, for any vertex a of E, $E(a)$ is equal as an element of $TL(P, \mathcal{C})$ to a heap with property P2;*

(iii′) *whenever $p_1 p_2 \cdots p_r$ is a monomial basis element of $TL(P, \mathcal{C})$, the monomial obtained by deleting one of the p_i is equal (as an element of $TL(P, \mathcal{C})$) to a monomial basis element.*

FIGURE 1. Connected graphs associated with regular classes of heaps



The classification of regular classes of heaps is in terms of their concurrency graphs (see Definition 1.1.5). This involves the graphs in Figure 1. For types A_n, D_n and E_n, n is the number of nodes in the graph, and we assume $n \geq 4$ for type D_n and $n \geq 6$ for type E_n to avoid repetition. (Note that we require graphs E_n for arbitrarily high n.) The graph of type $\widetilde{A}_{n-1}$ has n nodes where n is odd and $n \geq 3$. The graph of type $\widetilde{E}_6$ has exactly 7 nodes.

Theorem 1.5.2. *Let $H(P, \mathcal{C})$ be the class of heaps with pieces in P and concur-*

rency relation $\mathcal{C}$. Suppose also that P is a finite set, and let Γ be the concurrency graph associated to $H(P, \mathcal{C})$. Then $H(P, \mathcal{C})$ is a regular class of heaps if and only if each connected component of Γ is a complete graph K_n or appears in the list depicted in Figure 1: type A_n $(n \geq 1)$, type D_n $(n \geq 4)$, type E_n $(n \geq 6)$, type $\widetilde{A}_{n-1}$ $(n \geq 3$ and n odd$)$ or type $\widetilde{E}_6$.

Remark 1.5.3. We assume that P is finite above to avoid cardinality considerations.

2. Characterization of regular classes of heaps

The aim of §2 is to prove Theorem 1.5.1. In order to do so we shall need to recall and develop some additional combinatorial properties of heaps.

2.1 Factorization of heaps.

Heaps possess the following unique factorization property.

Proposition 2.1.1. *Any heap $E \in H(P, \mathcal{C})$ can be written uniquely as a product of trivial heaps*

$$E = T_1 \circ T_2 \circ \cdots \circ T_p$$

such that for each $1 \leq j < p$ and for each $b \in T_{j+1}$, there exists $a \in T_j$ with $a < b$.

Note. Note that, in the above situation, it is possible for $\varepsilon(a) = \varepsilon(b)$.

Proof. See [**13**, Lemma 2.9]. □

Example 2.1.2. The factorization of the heap E arising from Example 1.1.2 is of the form $T_1 \circ T_2 \circ T_3$ whose underlying sets are given by $T_1 = \{a, b\}$, $T_2 = \{c\}$ and $T_3 = \{d, e\}$.

Remark 2.1.3. In the unique factorization of the heap E given by Proposition 2.1.1, the heap T_1 is the subheap of E consisting of the vertices $\mathcal{L}(E)$. (See the proof of [**13**, Lemma 2.9] for justification, and Definition 1.2.2 for the notation.) It is not always the case that the rightmost subheap in the factorization of E coincides with $\mathcal{R}(E)$ (although it does in Example 1.2.3).

Definition 2.1.4. Let $E = [E, \leq, \varepsilon]$ be a heap of $H(P, \mathcal{C})$ and let

$$E = T_1 \circ T_2 \circ \cdots \circ T_p$$

be its unique factorization as in Proposition 2.1.1. We define $E^* = [E, \geq, \varepsilon]$, the *opposite heap* of E, to be the heap

$$T_p \circ T_{p-1} \circ \cdots \circ T_1.$$

If a is a vertex of E situated in the factor T_i then we denote the corresponding element of E^* in the factor T_i by a^*. (Note that $\varepsilon(a^*) = \varepsilon(a)$.)

We define the *double*, $\Delta(E)$, of E to be the heap of $H(P, \mathcal{C})$ given by

$$\Delta(E) = T_p \circ T_{p-1} \circ \cdots \circ T_2 \circ T_1 \circ T_2 \circ \cdots \circ T_{p-1} \circ T_p.$$

Note that $\Delta(E)$ contains both E and E^* as subheaps.

Remark 2.1.5. The expression given above for $\Delta(E)$ in terms of trivial heaps may or may not be the unique factorization of $\Delta(E)$ given by Proposition 2.1.1.

Example 2.1.6. Consider the heap of Example 1.1.2. In this case, the opposite heap E^* is given by

$$(d^* \circ e^*) \circ (c^*) \circ (a^* \circ b^*),$$

where the parentheses enclose the factors in the unique factorization of E^*. The double of E is given by

$$\Delta(E) = (d^* \circ e^*) \circ (c^*) \circ (a \circ b) \circ (c) \circ (d \circ e),$$

which happens to be the factorization of $\Delta(E)$ given by Proposition 2.1.1.

2.2 More on the relation $\prec$.

For the main results of §2, we will need to examine more closely the relation $\prec$ that appears in the definition of property P1. Interesting examples of these results will be found in §3 when we classify the regular classes of heaps.

The next result is used in [**3**, §4].

Lemma 2.2.1. *Let $E = [E, \leq, \varepsilon]$ be a heap with unique factorization (as in Proposition 2.1.1)*

$$E = T_1 \circ T_2 \circ \cdots \circ T_p$$

where $p > 1$, and suppose there is no subheap F of E with $F \prec^- E$. Then for each vertex $b \in T_2$, either

(i) *there is a vertex $a \in T_1$ such that $\varepsilon(a) = \varepsilon(b)$, or*

(ii) *there are at least two vertices $a_1, a_2 \in T_1$ such that $a_i < b$ and such that $\varepsilon(a_1), \varepsilon(a_2), \varepsilon(b)$ are distinct.*

Proof. Suppose that (i) does not hold. (Recall that the vertices in T_1 are precisely the minimal vertices of E.)

The existence of one vertex, a_1, satisfying the hypotheses is guaranteed by Proposition 2.1.1. Since (i) does not hold, $\varepsilon(a_1) \neq \varepsilon(b)$, which means that if there is no vertex a_2 with $\varepsilon(a_2) \neq \varepsilon(b)$ that is distinct from $a_1 \in T_1$, we must have $b \in \mathcal{L}(E(a_1)) \backslash \mathcal{L}(E)$ and $E(a_1) \prec^- E$, a contradiction. Finally, $\varepsilon(a_1) \neq \varepsilon(a_2)$, because there cannot be a chain $a_1 < a_2 < b$ unless $b \in T_i$ for $i \geq 3$. □

Lemma 2.2.2. *Maintain the notation of Definition 1.2.4. Let $E = [E, \leq, \varepsilon]$ be a heap of $H(P, \mathcal{C})$ containing no balanced convex chains of length 2. Suppose that there is no subheap F of E with $F \prec^+ E$.*

(i) *If $E(a) \prec^- E$, then $\Delta(E) = \delta^m \Delta(E(a))$ as elements of $TL(P, \mathcal{C})$, for some nonnegative integer m. If, in addition, $E(a)$ is trivial, we have $m \geq 1$.*

(ii) *If a is a minimal element of E (in particular, if $E(a) \prec^- E$) then there is no subheap F' of $E(a)$ with $F' \prec^+ E(a)$.*

Proof. We first prove (ii). Definition 1.2.4 implies that if $F' \prec^+ E(a)$ then $a \circ F' \prec^+ E$: by hypothesis, a does not interfere with the properties of maximal elements required by the relation $\prec^+$, proving (ii).

Suppose now that $E(a) \prec^- E$ as in the statement of (i), so that there is a vertex $b \in \mathcal{L}(E(a)) \backslash \mathcal{L}(E)$ with $\varepsilon(b) \neq \varepsilon(a)$. We denote the (possibly empty) set

$\mathcal{L}(E(a))\backslash(\mathcal{L}(E) \cup \{b\})$ by M. Let $T_1 \circ T_2 \circ \cdots \circ T_p$ be the unique factorization of the heap $E(a)$.

Now $b^* \circ a \circ b$ gives a balanced convex chain, $\mathbf{c}$, of length 3 in the double $\Delta(E)$ of E. Since $\varepsilon(b^*) = \varepsilon(b) \neq \varepsilon(a)$, Lemma 1.4.5 shows that $\Delta(E) = \Delta(E)/\mathbf{c}$ in $TL(P, \mathcal{C})$. The heap $\Delta(E)/\mathbf{c}$ is closely related to $\Delta(E(a))$; the only difference is that each element μ of T_1 in $\Delta(E(a))$ corresponding to an element of the set M is replaced in $\Delta(E)/\mathbf{c}$ by two elements, $\mu^* \circ \mu$. These correspond to balanced convex chains of length 2 and we may apply Lemma 1.4.5 again to show that $\Delta(E) = \delta^m \Delta(E(a))$, where $m = |M|$.

It remains to consider the case where $E(a)$ is trivial; this implies that E is not trivial because $E(a) \prec^- E$. In this case, $E = a \circ E(a)$. Let $T_1' \circ T_2'$ be the unique factorization of E; then the vertex a lies in the factor T_1'. Let $T_1'' \circ T_2''$ be the unique factorization of the opposite heap, E^* of E; the factor T_2'' contains only the vertex a^*. Since (by hypothesis) there is no subheap F of E with $F \prec^+ E$, it follows that there is no subheap F of E^* with $F \prec^- E^*$. Lemma 2.2.1 applies to the heap E^*, but case (i) cannot occur because E (and therefore E^*) contains no balanced convex chains of length 2. There are therefore at least two vertices b_i^* in E^* such that $b_i^* < a^*$, meaning that $b_i > a$ for each such vertex. The vertices $b_i \in E$ all lie in the factor T_2', which is a trivial subheap. Let M' be the set of all such vertices b_i.

Iterated applications of Lemma 1.4.5 show that

$$\left(\prod_{b \in M'} b^*\right) \circ a \circ \left(\prod_{b \in M'} b\right) = \delta^{|M'|-1} \left(\prod_{b \in M'} b\right)$$

as elements of $TL(P, \mathcal{C})$. (See Remark 1.1.7 for the notation.) Since the elements of E that do not appear in the above product (a) commute with the elements that do appear and (b) appear in the factor T_1', we have

$$\Delta(E) = \delta^{|M'|-1} \Delta(E(a)),$$

and the conclusion of (i) follows from the fact that $|M'| \geq 2$. $\square$

Lemma 2.2.3. *Maintain the above notation. Suppose that $E = [E, \leq, \varepsilon]$ is a heap of $H(P, \mathcal{C})$ with property P1 and that there is no subheap F of E with $F \prec^+ E$. Then:*

(i) *there is a sequence*

$$E_1 \prec^- E_2 \prec^- \cdots \prec E_{r-1} \prec^- E_r = E$$

of heaps with E_1 trivial, and for each E_i there is no subheap F_i of E_i with $F_i \prec^+ E_i$;

(ii) *$\Delta(E) = \delta^m E_1$ as elements of $TL(P, \mathcal{C})$ for some nonnegative integer m, and $m > 0$ if E is nontrivial;*

(iii) *if a is a minimal element of E then there is a sequence like that of (i) but with final term $E_r = E(a)$; in particular, $E(a)$ has property P1.*

Proof. Part (i) is a consequence of the definition of property P1 and Lemma 2.2.2 (ii), whose hypotheses are satisfied by Theorem 1.3.3 (b).

For part (ii), note that $E_1 = \Delta(E_1)$ because E_1 is trivial. The claim follows from (i) and repeated applications of Lemma 2.2.2 (i); if E is nontrivial, we have $m > 0$ by considering $E_1 \prec^- E_2$.

We now turn to (iii). The steps $E_{i-1} \prec^- E_i$ in the sequence in (i) correspond to the removal of a sequence of (distinct) vertices a_i, each minimal in E_i. According to the definition of $\prec^-$, the removal of each vertex a_i exposes a new minimal vertex b_i that is not minimal in E_i. Since a is minimal in E we cannot have $a_i < a$ for any of the elements a_i.

There are now two cases to consider. In the first case, a is not equal to any of the a_i. In this case, we can adapt the sequence in (i) by replacing each E_i by $E_i(a)$, and the required properties hold. In the second case, a is equal to one of the a_i. In this case, the sequence

$$E_1 \prec^- E_2 \prec^- \cdots \prec^- E_{i-1} = E_i(a) \prec^- E_{i+1}(a) \prec^- \cdots \prec^- E_r(a) = E(a)$$

has the required properties. $\square$

2.3 More on doubles of heaps.

Lemma 2.3.1. *If $E = [E, \leq, \varepsilon]$ is a heap of $H(P, \mathcal{C})$ with property P2 but not property P1, then $\Delta(E)$ has property P2.*

Proof. It is enough to show that if $x < z$ are two vertices of $\Delta(E)$ with $\varepsilon(x) = \varepsilon(z)$, then there exist at least two distinct vertices y_1 and y_2 with $x < y_1 < z$, $x < y_2 < z$ and $\varepsilon(y_i) \neq \varepsilon(x)$ for $i \in \{1, 2\}$. We may assume that there is no element w with $x < w < z$ and $\varepsilon(x) = \varepsilon(w) = \varepsilon(z)$, or E would have a balanced convex chain of length 2.

Let us write

$$\Delta(E) = T_p \circ T_{p-1} \circ \cdots \circ T_2 \circ T_1 \circ T_2 \circ \cdots \circ T_{p-1} \circ T_p.$$

Note that $p > 1$ because E cannot be trivial.

If x and z both come from factors in or to the right of the factor T_1, then we are done because E has property P2. Similarly, if x and z both come from factors in or to the left of the factor T_1, we are done because E^* inherits property P2 from E. We may therefore assume that neither x nor z comes from T_1, that x comes from a factor to the left of T_1, and that z comes from a factor to the right of T_1. The assumptions made above regarding the element w imply that $x = z^*$.

Suppose that z comes from the factor T_i with $i > 2$. This means that z is not minimal in E. By Proposition 2.1.1, there is an element y in the factor T_{i-1} with $y < z$ and $\varepsilon(y) \neq \varepsilon(z)$ (since E has property P2), and we may take $y_1 = y$ and $y_2 = y^*$ (note that $y^* \neq y$ because $i > 2$).

The other case is when z comes from the factor T_2. In this case we apply Lemma 2.2.1 with $b = z$. Case (i) cannot hold, because it would contradict the assumption that E has property P2. We then take $y_i = a_i$ for $i \in \{1, 2\}$. □

Proof of Theorem 1.5.1. The equivalence of (ii) and (ii′) of Theorem 1.5.1 follows from Theorem 1.3.3 (a). The equivalence of (iii) and (iii′) follows from Definition 1.4.7.

The implication (i) $\Rightarrow$ (ii) comes from [**9**, Theorem 2.4.2] (see Theorem 1.3.3 (c) (i)) for the statement).

Next consider the implication (ii) $\Rightarrow$ (iii). By [**9**, Theorem 3.2.3], we see that $E(a) = \delta^m G$ in $TL(P, \mathcal{C})$, for some heap G with property P2 and some nonnegative integer m. Furthermore, that result shows that $\dim \ker \partial_{E(a)} = m + \dim \ker \partial_G$. Since, by (ii), G is (strongly) acyclic, we have $\dim \ker \partial_{E(a)} = m$. Since E has property P2, it is strongly acyclic and $E(a)$ is therefore acyclic, forcing $m = 0$ and proving the claim.

We will be done if we can show that the negation of (i) implies the negation of (iii). Let E be a heap with property P2 but not property P1, and suppose that E is minimal with this property, in the sense that whenever a is a maximal or minimal vertex of E then $E(a)$ has property P1. (Note that $E(a)$ will also have property P2 in this situation.) Any such heap E is necessarily nontrivial, so by Lemma 2.3.1, $\Delta(E)$ also has property P2.

Let a be a minimal element of E. The minimality property of E shows that $\mathcal{L}(E) = \mathcal{L}(E(a)) \cup \{a\}$, which implies that $\Delta(E(a)) = \Delta(E) \backslash \{a\}$ (embedding E in $\Delta(E)$ in the usual way). Furthermore, there is no subheap F of $E(a)$ with $F \prec^+ E(a)$ by Lemma 2.2.2 (ii), and $E(a)$ has property P1 by minimality of E. Since any heap that has no balanced convex chains of length 2 and that is of the form $a \circ T$ for T trivial must have property P1, $E(a)$ is not a trivial heap. By Lemma 2.2.3 (ii), $\Delta(E(a)) = \delta^m E_1$ (as elements of $TL(P, \mathcal{C})$) for some $m > 0$ and some trivial heap E_1. This means that, as elements of $TL(P, \mathcal{C})$, $\Delta(E) \backslash \{a\} = \delta^m E_1$ for some $m > 0$, which contradicts (iii), completing the proof. $\square$

3. Classification of regular classes of heaps

In §3, we shall prove Theorem 1.5.2. The proof techniques we use are reminiscent of those used in the classification of finite Coxeter groups (see [**11**, §2]) and the classification of FC-finite Coxeter groups (see [**12**, §4] or [**5**, §7]).

3.1 Subgraphs and connected components.

Our classification of regular classes of heaps is in terms of their concurrency graphs (Definition 1.1.5).

Definition 3.1.1. A graph Γ is said to have *property R* if it is the concurrency graph of a regular class of heaps.

Our aim is to classify all finite graphs with property R. The key to the procedure is the following observation.

Lemma 3.1.2. *Let $H(P, \mathcal{C})$ be a class of heaps, let P' be a subset of P and let $\mathcal{C}'$ be the restriction of $\mathcal{C}$ to P'.*

(i) *There is a canonical inclusion, ι, of $H(P', \mathcal{C}')$ into $H(P, \mathcal{C})$ that respects the partial order on heaps and the labelling function $\varepsilon : E \longrightarrow P$.*

(ii) *The heap, E has property P1 (respectively, property P2) if and only if $\iota(E)$ does.*

Proof. Let $E = [E, \leq, \varepsilon]$ be a heap of $H(P', \mathcal{C}')$. The heap $\iota(E)$ of $H(P, \mathcal{C})$ is $[E, \leq, \eta]$, where $(E, \leq)$ is the same poset as before and η is the obvious extension of ε to $\mathcal{C}$. If a, b are elements of E, then $\varepsilon(a)\ \mathcal{C}'\ \varepsilon(b)$ if and only if $\eta(a)\ \mathcal{C}\ \eta(b)$, so both axioms of Definition 1.1.1 hold, proving (i).

The assertion of (ii) follows because properties P1 and P2 can be defined using only the partial order on the heap and the function ε. $\square$

Lemma 3.1.3. *If Γ' is a full subgraph of a graph Γ with property R, then Γ' has property R.*

Proof. If Γ is the concurrency graph of $H(P, \mathcal{C})$ and Γ' is a full subgraph of it, then Γ' must correspond to $H(P', \mathcal{C}')$ for some subset P' of P, where $\mathcal{C}'$ is the restriction of $\mathcal{C}$ to P'. If we denote the canonical embedding $H(P', \mathcal{C}') \hookrightarrow H(P, \mathcal{C})$ by ι, Lemma 3.1.2 (ii) shows that a heap $E = [E, \leq, \varepsilon]$ of $H(P', \mathcal{C}')$ has property P2 but not property P1 if and only if $\iota(E)$ does. The conclusion follows. $\square$

Our considerations reduce quickly to problems about connected graphs, thanks to the following result.

Lemma 3.1.4. *Let $H(P, \mathcal{C})$ be a class of heaps with concurrency graph Γ, and suppose that Γ is the disjoint union of two subgraphs, Γ_1 and Γ_2. Let P_1 and P_2 be the respective subsets of P, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the restrictions of $\mathcal{C}$ to P_1 and P_2 respectively.*

(i) *Any heap $E = [E, \leq, \varepsilon]$ of $H(P, \mathcal{C})$ may be written uniquely as the product*

$$E = \iota_{P_1}(E_1) \circ \iota_{P_2}(E_2) = \iota_{P_2}(E_2) \circ \iota_{P_1}(E_1),$$

where E_i is a heap in $H(P_i, \mathcal{C}_i)$ and $\iota_{P_i}(E_i)$ denotes the embedding of E_i into $H(P, \mathcal{C})$ as in Lemma 3.1.2 (i).

(ii) *The heap E has property P1 (respectively, property P2) if and only if E_1 and E_2 both have property P1 (respectively, property P2).*

Proof. For $i \in \{1, 2\}$, define E_i to be the heap of $H(P_i, \mathcal{C}_i)$ corresponding via the map ι_{P_i} to the subheap of E whose vertices are precisely those vertices a with $\varepsilon(a) \in P_i$. Observe that if a and b are vertices of E with $a \leq b$ and $\varepsilon(a) \ \mathcal{C} \ \varepsilon(b)$, we must have $a, b \in P_i$ for the same i, and $\varepsilon(a) \ \mathcal{C}_i \ \varepsilon(b)$. It follows from Definition 1.1.1 that the poset $(E, \leq)$ is the disjoint union of $(E_1, \leq_1)$ and $(E_2, \leq_2)$, where $\leq_i$ is the restriction of $\leq$ to E_i.

Claim (i) follows from the facts that (a) $k \circ l = l \circ k$ whenever it is not the case that $\varepsilon(k) \ \mathcal{C} \ \varepsilon(l)$ and (b) E can be written as a finite $\circ$-product of singleton heaps. For (ii), we may replace the heaps E_i by $\iota_{P_i}(E_i)$ by Lemma 3.1.2 (ii), and the claim follows by an argument similar to the last part of the proof of that result. $\square$

Corollary 3.1.5. *A graph Γ has property R if and only if all its connected components do.*

Proof. If Γ has property R then any connected component does by Lemma 3.1.3. If all the connected components have property R, then Γ does by Lemma 3.1.4 (ii). $\square$

3.2 Some graphs with property R.

In §3.2, we prove that all the graphs mentioned in Theorem 1.5.2 have property R. Much of the work for this has been done by Fan:

Theorem 3.2.1 (Fan). *If Γ is a graph of type A_n, D_n, E_n or $\widetilde{A}_{n-1}$ with n odd, then Γ has property R.*

Proof. For types A, D and E, this is a restatement of [**3**, Lemma 4.3.1], and for type $\widetilde{A}_{n-1}$, this is a restatement of [**4**, Proposition 3.1.2]. (See also [**9**, Theorem 3.4.1].) $\square$

Proposition 3.2.2. *If Γ is a complete graph, K_n, then Γ has property R.*

Proof. Let $H(P, \mathcal{C})$ be a class of heaps with concurrency graph Γ, and let $E = [E, \leq, \varepsilon]$ be a heap of $H(P, \mathcal{C})$ with property P2; we will show that E has property P1.

Let $T_1 \circ \cdots \circ T_p$ be the unique factorization of E as in Proposition 2.1.1. Since Γ is a complete graph, all the factors T_i must contain a single element, because the T_i are trivial. We may assume $p > 1$ (or we are done). Let a be the vertex in the factor T_1, and let b be the vertex in the factor T_2. Since E has property P2, we cannot have $\varepsilon(a) = \varepsilon(b)$, but we must have $\varepsilon(a) \ \mathcal{C} \ \varepsilon(b)$ because Γ is complete. We therefore have $E(a) \prec^- E$, and the claim follows by induction on $|E|$. $\square$

We introduce the following definition for notational convenience.

Definition 3.2.3. Let $E = [E, \leq, \varepsilon]$ be a heap of $H(P, \mathcal{C})$ with unique factorization

$$E = T_1 \circ T_2 \circ \cdots \circ T_p.$$

We say that the piece w is *represented* in the factor T_i by $a \in E$ if there is a vertex a in the factor T_i with $\varepsilon(a) = w$. We say that w *occurs* in the factor T_i if it is represented by some vertex a in T_i.

Example 3.2.4. In the heap of Example 1.1.2, the piece 3 occurs in T_1 and T_3; it is represented by b in T_1 and by e in T_3.

We now turn our attention to type $\widetilde{E}_6$. It is notationally convenient to assign names to the vertices of the graph (i.e., the pieces). We call the branch point c; its neighbours are b, d and f, and the other vertices adjacent to b, d and f are denoted by a, e and g, respectively. (In Figure 1, the labels could read a, b, c, d, e along the top row, and then f and g reading downwards.) We will consider a minimal counterexample to regularity in the sense of the proof of Theorem 1.5.1, that is, a heap E with property P2 but not P1, but such that any heap $E(a)$ for a maximal or minimal vertex a has property P1. (By an argument similar to that used to prove Lemma 2.2.3 (iii), $E(a)$ will automatically have property P2.)

Lemma 3.2.5. *Let $H(P, \mathcal{C})$ be a class of heaps with concurrency graph $\widetilde{E}_6$ and let E be a minimal heap with property P2 but not property P1 (in the sense of the above discussion). Let $T_1 \circ \cdots \circ T_p$ be the unique factorization of E. Then:*

(i) *$p \geq 3$;*

(ii) *c occurs in T_2;*

(iii) *b, d and f are represented by vertices $a_1, a_2 \in T_1$ and $a_3 \in T_3$, in some order, and $\varepsilon(a_3)$ does not occur in T_1.*

Proof. Clearly E cannot be trivial, because trivial heaps have property P1, so $p > 1$. If E has property P2 but not P1, then the opposite heap E^* does as well, and Lemma 2.2.1 (ii) makes it impossible to have $p = 2$ because Γ is a finite graph with no circuits. This proves (i).

Choose a minimal vertex, γ, of E. By the minimality of E, $E(\gamma)$ has property P1. By Lemma 2.2.2 (ii) and repeated applications of Lemma 2.2.3 (iii) (starting with the heap $E(\gamma)$), we find that there exists a heap F with $F \prec^- E'$, where

$$E' = T_2 \circ T_3 \circ \cdots \circ T_p.$$

Let $\alpha \in T_2$ and $\beta \in T_3$ be as in the definition of the condition $F \prec^- E'$. If κ were to represent $\varepsilon(\beta)$ in T_1 then $\kappa < \alpha < \beta$ would be a balanced convex chain contradicting the assumption that E has property P2, so $\varepsilon(\beta)$ cannot occur in T_1.

However, Lemma 2.2.1 (ii) guarantees that at least two neighbours of $\varepsilon(\alpha)$ in the graph Γ occur in T_1, so the valency of $\varepsilon(\alpha)$ must be at least 3, i.e., $\varepsilon(\alpha) = c$, proving (ii). Part (iii) follows as a consequence of the same argument. $\square$

Proposition 3.2.6. *A graph of type $\widetilde{E}_6$ has property R.*

Proof. Let $H(P, \mathcal{C})$ be a class of heaps with concurrency graph $\widetilde{E}_6$ and suppose that E is a minimal heap with property P2 but not P1, as in the proof of Lemma 3.2.5. Let $T_1 \circ \cdots \circ T_p$ be the unique factorization of E.

By symmetry of Γ and Lemma 3.2.5, we may assume that b and d occur in T_1 and f occurs in T_3 but not T_1. The only other vertex that could occur in T_1 is g, since T_1 is trivial. In T_2, there cannot be any occurrences of a or e, because then removal of the occurrences of b or d in T_1 would violate the hypotheses on E. It follows that only c can occur in T_2, and $|T_2| = 1$. We have just established that the monomial of $\text{Co}(P, C)$ (where C is the complementary relation of $\mathcal{C}$) corresponding to $T_1 \circ T_2$ is either $bdgc$ or bdc.

We know that T_3 contains an occurrence of f. Any other elements of P occurring in T_3 must be adjacent to c by Proposition 2.1.1. However, b and d cannot occur because subwords $bdgcb$ and $bdgcd$ are not allowed in a heap with property P2, so $|T_3| = 1$. Both the sequences given correspond to heaps with property P1, so we must have $p > 3$.

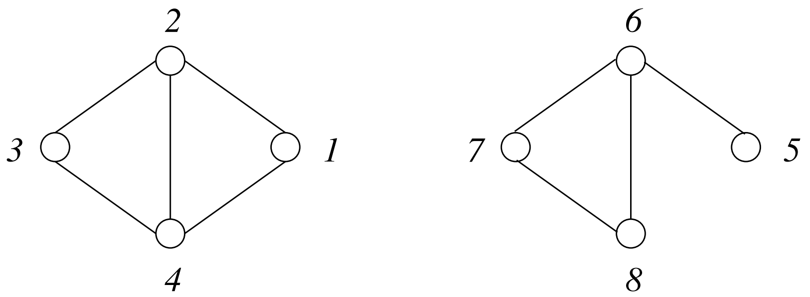
Using a similar argument, we find that the only element of P that can occur in T_4 is g, and even this is not allowed if g occurs in T_1, because E has property P2. The monomial of $\text{Co}(P, C)$ corresponding to E can therefore be assumed to start $bdcfg\ldots$, and this cannot be the complete monomial since the first 5 letters correspond to a heap with property P1. However, no further letters can be added to it on the right without violating one of the hypotheses. We conclude that no such heap E exists, and that Γ has property R. $\square$

3.3 Triangles, circuits and branch points.

All that remains to prove Theorem 1.5.2 is to classify the connected graphs that fail to have property R. Lemma 3.1.3 reduces this problem to a problem about complete subgraphs. However, Lemma 3.1.3 is false if we delete the word "full" (see Proposition 3.2.2). To get around this problem, we first consider the case where Γ contains a triangle.

Lemma 3.3.1. *If Γ is a graph with property R, then Γ has no full subgraphs of either of the types shown in Figure 2.*

FIGURE 2. Some 4-vertex graphs without property R



Proof. The two graphs shown in Figure 2 are the only connected graphs with four vertices that contain a triangle but are not the complete graph K_4.

Consider the commutation monoid $\mathrm{Co}(P, C)$ corresponding to the graph on the left. In this case, only the vertices 1 and 3 commute with each other, and the monomial $(13)(2)(4)(13)$ corresponds to a heap with property P2 but not property P1, so this graph does not have property R. More precisely, the corresponding heap is of the form $T_1 \circ T_2 \circ T_3 \circ T_4$ where each factor corresponds to a parenthetic expression in the monomial, so for example T_3 contains an occurrence of the piece 4, and no other vertices.

Now consider the graph on the right; this is a triangle on the vertices 6, 7 and 8, where 5 is connnected only to 6. Then the monomial $(58)(6)(7)(8)(6)(57)$ in the commutation monoid corresponds to a heap (with the indicated unique factorization) that has property P2 but not property P1. $\square$

Remark 3.3.2. A quick way to check that the monomials in the above proof have property P2 is to verify that any two occurrences of the same generator a are separated by at least two other occurrences of generators not commuting with a in $\mathrm{Co}(P, C)$. Fan calls this property R3; see [**3**, §2].

Lemma 3.3.3. *Let Γ be a finite connected graph containing a triangle, and suppose that every 4-vertex connected full subgraph of Γ that contains a triangle is the complete graph K_4. Then Γ is complete.*

Proof. If Γ has three vertices, there is nothing to prove, so we assume $|\Gamma| > 3$. Suppose Γ is not complete, and let Γ' be a maximal complete subgraph of Γ, which must therefore be a proper subgraph. Let a be a vertex of $\Gamma \backslash \Gamma'$ adjacent to some vertex b of Γ'; this must exist since Γ is connected. Let c be any other vertex of Γ'. Since $|\Gamma'| \geq 3$, there is a triangle in Γ' containing b, c and some other vertex, c'. Now $\{a, b, c, c'\}$ induces a 4-vertex connected full subgraph of Γ that contains a triangle, so it must be the complete graph. In particular, a is adjacent to c, and, since c was arbitrary, the full subgraph containing the vertices $\Gamma' \cup \{a\}$ is a complete subgraph properly containing Γ', a contradiction. □

Corollary 3.3.4. *If Γ is a finite connected graph with property R and Γ contains a triangle (as a subgraph), then Γ is complete.*

Proof. Suppose Γ is not complete. Then by Lemma 3.3.3, Γ contains one of the graphs in Figure 2 as a full subgraph. Lemma 3.3.1 and Lemma 3.1.3 then show that Γ does not have property R. □

Lemma 3.3.5. *If Γ is a finite incomplete connected graph with property R and Γ contains a circuit, then Γ is of type $\widetilde{A}_{n-1}$ for some odd number n.*

Proof. Let $g_1, g_2, \ldots, g_{l-1}, g_l = g_1$ be a circuit of vertices in Γ; note that $l > 3$. Assume that Γ is not itself a circuit. We may assume without loss of generality that g_1 is adjacent to some other vertex, x. Since Γ is incomplete, it contains no triangles by Corollary 3.3.4, so x is not adjacent either to g_2 or to g_{l-1}. The monomial

$$(xg_{l-1})(g_1)(g_2)\cdots(g_{l-2})(g_{l-1})(g_1)(xg_2)$$

in the commutation monoid associated to Γ then corresponds to a heap with property P2 but not property P1. This shows that Γ is a circuit.

If Γ is an even circuit, $g_1, g_2, \ldots, g_{2l}, g_{2l+1} = g_1$, then the monomial

$$(g_1 g_3 g_5 \cdots g_{2l-1})(g_2 g_4 g_6 \cdots g_{2l})$$

corresponds to a heap with property P2 but not property P1. This proves that n is odd. $\square$

Lemma 3.3.6. *If Γ is a finite incomplete connected graph with property R then Γ contains at most one branch point.*

Proof. Suppose Γ contains two branch points, c and c'. Then Γ is not a circuit, and it cannot contain a circuit by Lemma 3.3.5. Since Γ is connected, it contains a shortest path

$$c = g_1, g_2, \ldots, g_{k-1}, g_k = c'.$$

There are two vertices x_1 and x_2 distinct from g_2 that are adjacent to c, and similarly there are two vertices y_1 and y_2 distinct from g_{k-1} that are adjacent to c'. There are no coincidences among the g_i, x_i and y_i, because Γ contains no circuits. For the same reason, x_1 is not adjacent to x_2, and y_1 is not adjacent to y_2. Then the commutation monoid associated to Γ contains the monomial

$$(x_1 x_2)(g_1)(g_2)\cdots(g_{k-1})(g_k)(y_1 y_2),$$

which corresponds to a heap with property P2 but not property P1. $\square$

Lemma 3.3.7. *If Γ is a finite incomplete connected graph with property R then every vertex of Γ has valency strictly less than 4.*

Proof. Suppose that c is a vertex with valency 4 or greater, and let x_1, x_2, x_3, x_4 be four distinct vertices adjacent to c. By Corollary 3.3.4, none of the x_i is adjacent to any other. The monomial

$$(x_1x_2)(c)(x_3x_4)$$

in the commutation monoid corresponds to a heap with property P2 but not property P1. □

3.4 The graphs $\Gamma(p, q, r)$.

Definition 3.4.1. Let p, q, r be nonnegative integers with $p \leq q \leq r$. We define $\Gamma(p, q, r)$ to be the graph with $p + q + r + 1$ vertices containing one vertex, c, of valency 3 and disjoint arms of lengths p, q and r emanating from c.

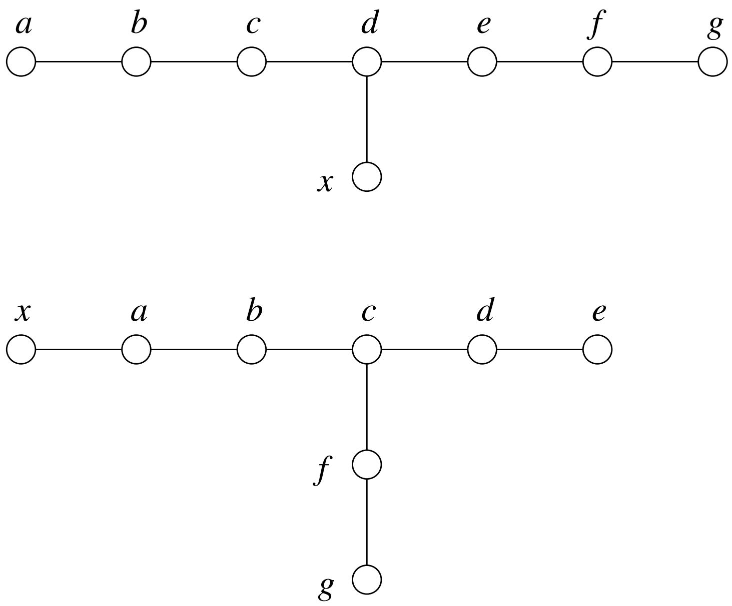
Example 3.4.2. In Figure 1, the graph of type A_n is $\Gamma(0, 0, n-1)$, the graph of type D_n is $\Gamma(1, 1, n-3)$ and the graph of type E_n is $\Gamma(1, 2, n-4)$.

Lemmas 3.3.5, 3.3.6 and 3.3.7 have the following consequence.

Corollary 3.4.3. *If Γ is a finite connected graph with property R then either Γ is complete, or Γ is an n-gon for some odd n, or $\Gamma = \Gamma(p, q, r)$ for some p, q and r.* □

Lemma 3.4.4. *If $\Gamma(p, q, r)$ has property R, then $q < 3$.*

FIGURE 3. The graphs $\Gamma(1,3,3)$ and $\Gamma(2,2,3)$



Proof. By Lemma 3.1.3, it is enough to show that $\Gamma(1,3,3)$ (also known as $\widetilde{E}_7$) fails to have property R. Label the graph as shown in Figure 3.

The monomial $(acx)(bd)(ce)(df)(xeg)$ in the commutation monoid corresponds to a heap with property P2 but not P1, and establishes the claim. $\square$

Lemma 3.4.5. *If $\Gamma(p,q,r)$ has property R and $p=q=2$, then $r=2$.*

Proof. By Lemma 3.1.3, it is enough to show that $\Gamma(2,2,3)$ fails to have property R. Label the graph as shown in Figure 3.

The monomial $(xbf)(ac)(bd)(ce)(df)(cg)(bf)(ac)(xbd)$ in the commutation monoid corresponds to a heap with property P2 but not P1, completing the proof. $\square$

Proof of Theorem 1.5.2. We need only consider the case where Γ is connected by Corollary 3.1.5. In the light of Theorem 3.2.1 and Corollary 3.4.3, we only need to check that the claim holds for $\Gamma = \Gamma(p,q,r)$. If the hypotheses of Lemma 3.4.5 hold, then we have type $\widetilde{E}_6$. If $p=0$, we are in type A_n. Otherwise, if $q=2$, we have $p=1$, and we are in type E_n. By Lemma 3.4.4, the only other possibility is $p=q=1$, giving type D_n. $\square$

4. Concluding remarks

It might be interesting to investigate the representation theory of the algebras $TL(P, \mathcal{C})$ in the case where the concurrency graph has property R. This was done by Fan in [**3**] in the ADE case, and there are several papers on the case of type $\widetilde{A}_{n-1}$, including [**7**] and [**6**]. More specifically, it would be interesting to know if the basis of monomials for $TL(P, \mathcal{C})$ (Definition 1.4.7) is always a tabular basis in the sense of [**8**]; this is true in type ADE by [**8**, Theorem 4.3.5], and in type $\widetilde{A}_{n-1}$ for n odd by [**8**, Theorem 6.4.8].

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