ON REPRESENTATIONS OF AFFINE TEMPERLEY–LIEB ALGEBRAS

R.M. Green

Department of Mathematics and Statistics
Lancaster University
Lancaster LA1 4YF
England
E-mail: r.m.green@lancaster.ac.uk

Abstract. We study the finite-dimensional simple modules, over an algebraically closed field, of the affine Temperley–Lieb algebra corresponding to the affine Weyl group of type A. These turn out to be closely related to the simple modules for a certain q-analogue of the annular algebra of V.F.R. Jones.

1. Introduction

The Temperley–Lieb algebra is a finite-dimensional algebra which was introduced in [**6**] and has been extensively studied in many papers, for example [**7**].

In [**1**], the author and C.K. Fan introduced a diagram calculus for the "affine" Temperley–Lieb algebra, an infinite-dimensional algebra which is related to affine Weyl group $W(\widehat{A}_l)$ of type A in the same way as the (ordinary) Temperley–Lieb algebras are related to the group algebras of the symmetric groups. One would like to be able to classify the finite-dimensional irreducible representations for this algebra, partly because this gives irreducible representations of the affine Hecke algebra $\mathcal{H}(\widehat{A}_l)$ and of the corresponding Weyl group $W(\widehat{A}_l)$. The "even rank" case (where $l+1$, the number of nodes in the extended Dynkin diagram, is even) is already well understood (see [**4**], [**5**]).

The diagram calculus in [**1**] is reminiscent of the diagram calculus of certain "annular" algebras which were introduced in [**3**], and which (following [**2**]) we call Jones algebras. It is natural to wonder whether there is a strong connection between them, especially as both are essentially quotients of the affine Hecke algebra.

This paper sheds light on these problems and explains how they are related. We will always work in the category of finite-dimensional modules for the affine Temperley–Lieb algebra, so the term "irreducible module" should be understood to mean "finite-dimensional irreducible module".

After reviewing, in §2, the diagram calculus of [**1**], we introduce q-analogues of the Jones algebras in §3. (The algebras of [**3**] give rise to the case $q=1$.) These q-Jones algebras (and certain subalgebras of them) turn out to be cellular algebras in the sense of [**2**], in the same way as the ordinary Jones algebras were cellular

The author was supported in part by an E.P.S.R.C. postdoctoral research assistantship.

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(see [**2**, §6]). This means that we can easily classify their irreducible modules over an algebraically closed field. In §4 we explain the connection between the objects of §2 and those of §3. This means that "most" of the irreducible modules for the affine Temperley–Lieb algebras are the same as those described in §3, under certain natural identifications.

This method deals with all the modules except those in the most dominant "cell" of the algebra in even rank. These do not behave in the same way, and we do not deal with them here. The reader is referred to [**4**, §4.3.1], where these modules are considered.

2. Affine Temperley–Lieb Algebras

The aim of §2 is to review the definition of the affine Temperley–Lieb algebra as given in [**1**, §4], and to explain how it is related to certain other infinite-dimensional algebras given by a calculus of diagrams.

2.1 The algebra D_n.

It turns out for our purposes to be convenient (for various reasons which will become clear) to work with an algebra of diagrams which contains the Temperley–Lieb algebra as a proper subalgebra. We start this section by defining this bigger algebra.

First, we recall the definition of an affine n-diagram from [**1**, §4.1].

Definition 2.1.1.

An affine n-diagram, where $n \in \mathbb{Z}$ satisfies $n \geq 3$, consists of two infinite horizontal rows of nodes lying at the points $\{\mathbb{Z} \times \{0, 1\}\}$ of $\mathbb{R} \times \mathbb{R}$, together with certain curves, called edges, which satisfy the following conditions:

(i) *Every node is the endpoint of exactly one edge.*
(ii) *Any edge lies within the strip $\mathbb{R} \times [0, 1]$.*
(iii) *If an edge does not link two nodes then it is an infinite horizontal line which does not meet any node. Only finitely many edges are of this type.*
(iv) *No two edges intersect each other.*
(v) *An affine n-diagram must be invariant under shifting to the left or to the right by n.*

We will identify any two diagrams which are isotopic to each other, so that we are only interested in the equivalence classes of affine n-diagrams up to isotopy. This has the effect that the only information carried by edges which link two nodes is the pair of vertices given by the endpoints of the edge. Later on in the paper we will consider "annular involutions" which allow the diagrams to be described without the use of curves, but the geometric viewpoint is often helpful.

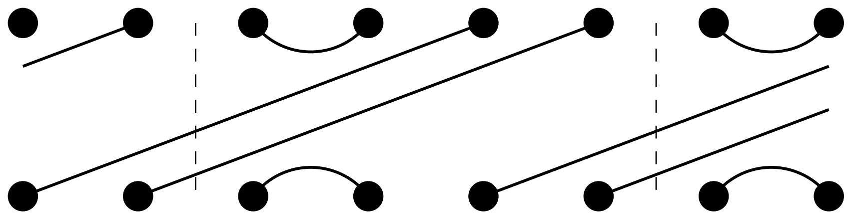
Another way to describe the equivalence of diagrams is as follows. If D_1 and D_2 are two affine n-diagrams and there is a bijection ϕ between the edges of D_1 with the edges of D_2 such that the set of endpoints of each edge $E \in D_1$ (possibly empty) is equal to the set of endpoints of the edge $\phi(E)$ in D_2 (possibly empty), then we say that D_1 and D_2 give the same graph, and we identify the diagrams.

Because of the condition (v), one can also think of affine n-diagrams as diagrams on the surface of a cylinder, or within an annulus, in a natural way. Unless otherwise specified, we shall henceforth regard the diagrams as diagrams on the surface of a cylinder with n nodes on top and n nodes on the bottom. Later, we will compare this with Jones' (finite-dimensional) annular algebra, which was introduced in [**3**]. From now on, we will call the affine n-diagrams "diagrams" for short, when the context is clear. Under this construction, the top row of nodes becomes a circle of n nodes on one face of the cylinder, which we will refer to as the top circle. Similarly, the bottom circle of the cylinder is the image of the bottom row of nodes.

The cylindrical viewpoint allows one to see why there is an infinite number of affine n-diagrams: edges which connect nodes on the top circle to nodes on the bottom circle may wind round the cylinder an arbitrary number of times, and different winding numbers give nonequivalent diagrams. We note that the idea of diagrams about a cylinder is familiar from Potts models in statistical mechanics (see the "boundary diagrams" in [**4**]).

An example of an affine n-diagram for $n = 4$ is given in Figure 1. The dotted lines denote the periodicity, and should be identified to regard the diagram as inscribed on a cylinder.

FIGURE 1. An affine 4-diagram



Definition 2.1.2. *An edge of the diagram D is said to be vertical if it connects a point in the top circle of the cylinder to a point in the bottom circle, and horizontal if it connects two points in the same circle of the cylinder.*

Two diagrams, A and B "multiply" in the following way, which was described in [**1**, §4.2]. Put the cylinder for A on top of the cylinder for B and identify all the points in the middle row. This produces a certain (natural) number x of loops. Removal of these loops forms another diagram C satisfying the conditions in Definition 2.1.1. The product AB is then defined to be $[2]^x C$, where $[2]$ is the Laurent polynomial $v + v^{-1}$ in the indeterminate v. It is clear that this defines an associative multiplication.

Definition 2.1.3. *Let R be a ring containing an invertible indeterminate v. We define the associative algebra D_n over R to be the R-linear span of all the affine n-diagrams, with multiplication given as above.*

2.2 The affine Temperley–Lieb algebra.

We assume from now on that $n \geq 3$, so there are at least three nodes in each circle of the cylinder.

Definition 2.2.1. *Let R be a commutative ring (with identity) containing* $[2]$*. The affine Temperley–Lieb algebra $TL(\widehat{A}_{n-1})$ is the R-algebra given by generators $E_1, \ldots, E_n$ and defining relations*

$$\begin{aligned} E_i^2 &= [2]E_i, && (1)\\ E_iE_j &= E_jE_i, \quad \text{if } \bar{i} \neq \overline{j \pm 1}, && (2)\\ E_iE_{\overline{i\pm 1}}E_i &= E_i. && (3) \end{aligned}$$

Here, $\bar{i}$ denotes the congruence class modulo n of i.

Remark 2.2.2. If n is even, and R is a field of characteristic 0, this agrees with the "periodic Temperley–Lieb algebra" studied in [**4**], although our notation is different. The generator U_i in [**4**] corresponds to our generator E_{2i-1}, and the generator U_{ij} (where $j = i + 1$) in [**4**] corresponds to our generator E_{2i}.

The algebra $TL(\widehat{A}_{n-1})$ occurs as a quotient of the affine Hecke algebra of type $\widehat{A}$ with algebra generators $T_1, T_2, \ldots T_n$. The generator E_i is the image of $v^{-1}T_e + v^{-1}T_i$, where T_e is the identity in the Hecke algebra, and the parameter q in the Hecke algebra is identified with v^2.

It was shown in [**1**, §4] that $TL(\widehat{A}_{n-1})$ can also be realised as an algebra of diagrams, as follows.

Proposition 2.2.3. *The algebra $TL(\widehat{A}_{n-1})$ is the subalgebra of D_n spanned by diagrams, D, with the following additional properties:*

(i) *If D has no horizontal edges, then D is the identity diagram, in which point j in the top circle of the cylinder is connected to point j in the bottom circle for all j.*

(ii) *If D has at least one horizontal edge, then the number of intersections of D with the line $x = i + 1/2$ for any integer i is an even number.*

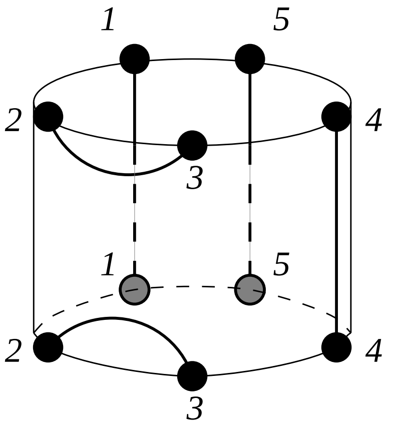
The identification sends the generator E_i to the diagram where there is a horizontal edge of minimal length connecting i and $\overline{i+1}$ in each of the circles of the cylinder, and there is a vertical edge connecting j in the top circle to j in the bottom circle whenever $j \neq i, \overline{i+1}$.

Proof. This follows from [**1**, Corollary 4.5.1]. □

Remarks. The number of intersections of D with a line $x = i + 1/2$ is, strictly speaking, not well defined, because edges may "loop back on themselves". We define the "number of intersections" as (in (ii)) to be the minimal number possible, which is obtained by perturbing the diagram of D so that each edge crosses the line as few times as possible. Precise details may be found in [**1**, §4.4].

A typical example of the representation of a generator E_i as a diagram on a cylinder is shown in Figure 2.

FIGURE 2. The element E_2 for $n = 5$

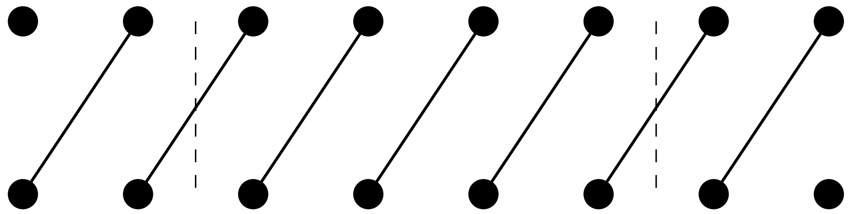


We now define an important element $u \in D_n$, similar to the element u of Jones' annular algebra which was considered in [**3**].

Definition 2.2.4. *The diagram u of D_n is the one satisfying the property that for all $j \in \mathbf{n}$, the point j in the bottom circle is connected to point $\overline{j+1}$ in the top circle by a vertical edge taking the shortest possible route.*

In the case $n = 4$, the element u is as shown in Figure 3.

FIGURE 3. The element u for $n = 4$



2.3 The algebra O_n.

We also introduce a subalgebra O_n of D_n. This is related to D_n in the same way as the alternating groups are related to the symmetric groups. (The letter "O" conceptually stands for "oriented", which is terminology borrowed from the paper [**3**].)

Recall from the remarks following Proposition 2.2.3 that the number of intersections of a diagram D with a line $x = i + 1/2$ refers to the minimum possible such number (since we are allowed to replace D by one of its isotopic images). In the results which follow, we are usually only interested in whether this number is even or odd, in which case it is not necessary that we take the minimum possible number of intersections, provided that any edges tangent to the line $x = i + 1/2$ are not counted as intersections.

Definition 2.3.1. *Define O_n to be the R-submodule of D_n spanned by diagrams D such that the number of intersections of D with the line $x = i+1/2$ for any integer i is an even number.*

Such diagrams are said to have the even intersection property. Conversely, if all the numbers of intersections of a diagram D with the lines $x + 1/2$ are odd, D is said to have the odd intersection property.

Lemma 2.3.2. *The following identity holds in D_n, regarding the generators E_i of $TL(\widehat{A}_{n-1})$ as elements of D_n:*

$$\underbrace{E_{\overline{i+2}}E_{\overline{i+3}}\cdots E_{\overline{i}}}_{n-1 \text{ terms}} = u^2 E_i.$$

Proof. Since $uE_iu^{-1} = E_{\overline{i+1}}$, it suffices to check the case $i = 1$. This follows easily from the diagrammatic viewpoint. □

Lemma 2.3.3. *The R-module O_n is the subalgebra of D_n generated by $TL(\widehat{A}_{n-1})$ and the elements $\{u^{2m} : m \in \mathbb{Z}\}$.*

Proof. It is not hard to see that if a diagram has no horizontal edges, it must be a power of u, and that the diagram lies in O_n if and only if this power of u is even. The fact that there are no more diagrams in O_n now follows from Proposition 2.2.3.

To check that these diagrams span a subalgebra, it remains to prove that u^2E_i and $u^{-2}E_i$ lie in O_n for each i. The case of u^2E_i follows from Lemma 2.3.2, and the case of $u^{-2}E_i$ follows by an analogous argument. The assertion now follows. □

The following lemma shows that any diagram has the even or odd intersection property, according as it is an element of O_n or not.

Proposition 2.3.4. *Let D be a diagram in D_n. If $D \in O_n$ (respectively, $D \notin O_n$), then the number of intersections of D with the line $x = i + 1/2$ is an even (respectively, odd) number for each i.*

Proof. If $D \in O_n$, it is clear from Lemma 2.3.3 that D has the even intersection property, because it is true if D is an even power of u or an element of $TL(\widehat{A}_{n-1})$. The converse, that any diagram with the even intersection property lies in O_n, is equally clear.

Next, we observe that if D has the even (respectively, odd) intersection property, then $u.D$ has the odd (repsectively, even) intersection property. This follows by consideration of the product $u.D$ via diagrams.

By consideration of the fact that each point in the diagram D has exactly one edge emerging from it, and that this edge can either emerge towards the left, towards the right, or straight upwards, we see that any diagram D in D_n has either the odd intersection property or the even intersection property as mentioned in the statement.

The proof now follows. □

Lemma 2.3.5. *As O_n-bimodules, $D_n = O_n \oplus uO_n = O_n \oplus O_nu$.*

Proof. We observe from the proof of Proposition 2.3.4 that any diagram of D_n is either a diagram of O_n or of the form $u.D$ where D is a diagram of O_n. We also note that for a diagram D, $D \in O_n \Rightarrow u^{-1}Du \in O_n$. The proof follows from these facts. □

We now define an automorphism ε of D_n which fixes O_n.

Proposition 2.3.6. *The R-linear map $\varepsilon : D_n \to D_n$ is defined by its effect on the diagrams D as follows:*

$$\varepsilon(D) = \begin{cases} D & \text{if } D \in O_n; \\ -D & \text{if } D \notin O_n. \end{cases}$$

This map is an automorphism of D_n whose fixed points are the elements of O_n.

Thus we have $\varepsilon(u) = -u$, $\varepsilon(u^{-1}) = -u^{-1}$ and $\varepsilon(E_i) = E_i$ for all i.

Proof. From Lemma 2.3.5, we find that any diagram in $D_n \backslash O_n$ can be written (uniquely) in the form $u.E$ where E is a diagram of O_n. Conversely, elements of these forms do not lie in O_n. Using the fact that $u.O_n.u^{-1} = O_n$ and the fact that even powers of u lie in O_n, we see that the map ε as described respects the multiplication (because two diagrams in $D_n \backslash O_n$ multiply to give an element of O_n, and so on).

The effect of ε on the generators and the assertion about the fixed points of ε are easy to prove. □

To conclude this section, we note that D_n has a nice presentation in terms of generators and relations, as follows.

Proposition 2.3.7. *The algebra D_n is generated by elements*

$$E_1, \ldots, E_n, u, u^{-1}.$$

It is subject to the relations (1), (2) *and* (3) *of Definition 2.2.1, and the following additional defining relations:*

$$uE_iu^{-1} = E_{\overline{i+1}}, \tag{4}$$
$$(uE_1)^{n-1} = u^n.(uE_1). \tag{5}$$

Proof. From Lemma 2.3.3 and Lemma 2.3.5, we see that the given set does generate the algebra D_n. In addition, all the relations are true; (5) looks unfamiliar but we claim that it is equivalent to Lemma 2.3.2 after applications of (4). To do this, first reexpress (5) as $u^{-(n-1)}(uE_1)^{n-1} = u^2E_1$. Now apply (4) to quasi-commute all the occurrences u in the left hand side to the left of the expression. This establishes the special case of Lemma 2.3.2 where $i = 1$. To obtain the general expression, we conjugate each side of the equation by a suitable power of u.

We then observe that any word in the generators can be rewritten as a multiple of a power of u times a monomial in the E_i (i.e. a diagram of $TL(\widehat{A}_{n-1})$) by using the relations (1) to (4). Using Lemma 2.3.2 (i.e. relation (5)), one can reduce this power of u to 0 or 1, unless the word is simply a power of u. Lemma 2.3.5 now shows that the relations are sufficient. □

Remark 2.3.8. In fact, D_n is generated by u, u^{-1} and E_1, as can be seen from relation (4).

3. q-Jones Algebras

In §3, we assume that $K = F(v)$ is an algebraically closed field, where F is a subfield of K and v is an indeterminate. We then introduce a family of finite-dimensional algebras $J_q(n)$, where q is a nonzero element of K. We call $J_q(n)$ the q-Jones algebra of rank n, where n is a natural number at least 3. The behaviour of these algebras in odd and in even ranks turns out to be distinctly different; if n is odd then $J_1(n)$ is Jones's annular algebra, $J(n)$, as defined in [**3**]. However, if n is even, $J_1(n)$ is not the full Jones algebra but a quotient of the subalgebra spanned by the "oriented elements" of $J(n)$. In both even and odd case, the algebras turn out to be cellular in the sense of [**2**], which enables their simple modules over K to be easily determined. These q-Jones algebras appear to be new.

3.1 Cellular algebras.

For completeness, we recall the definition of a cellular algebra from [**2**].

Definition 3.1.1. *Let R be a commutative ring with identity. A cellular algebra over R is an associative unital algebra, A, together with a cell datum $(\Lambda, M, C, *)$ where*

1. *Λ is a poset. For each $\lambda \in \Lambda$, $M(\lambda)$ is a finite set (the set of "tableaux" of type λ) such that*
$$C : \coprod_{\lambda \in \Lambda} (M(\lambda) \times M(\lambda)) \to A$$
is injective with image an R-basis of A.
2. *If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, we write $C(S,T) = C^\lambda_{S,T} \in A$. Then $*$ is an R-linear involutory anti-automorphism of A such that $(C^\lambda_{S,T})^* = C^\lambda_{T,S}$.*
3. *If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ then for all $a \in A$ we have*
$$a.C^\lambda_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^\lambda_{S',T} \mod A(< \lambda),$$
where $r_a(S', S) \in R$ is independent of T and $A(< \lambda)$ is the R-submodule of A generated by the set
$$\{C^\mu_{S'',T''} : \mu < \lambda, S'' \in M(\mu), T'' \in M(\mu)\}.$$

3.2 Annular Involutions.

Let D be a diagram (as in §2) representing a basis element of the extended affine Temperley–Lieb algebra D_n. Throughout §3.2, we are only concerned with diagrams D with at least one vertical edge. We define a certain integer $w(D)$ as follows.

Definition 3.2.1. *Let D be as above. Let $w_1(D)$ be the number of pairs $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ where $i > j$ and $\overline{j}$ in the bottom circle of D is joined to $\overline{i}$ in the top circle of D by an edge which crosses the "seam" $x = 1/2$. We then define $w_2(D)$ similarly but with the condition that $i < j$, and we define $w(D) = w_1(D) - w_2(D)$.*

Remark 3.2.2. We think of $w(D)$ as measuring some kind of "winding number" of the diagram D.

The non-intersection condition guarantees that at least one of $w_1(D)$ or $w_2(D)$ is 0, and that $w(D)$ is always finite. However, the values of $w(D)$ for $D \in D_n$ are unbounded even though n is fixed.

Some examples of the values of $w(D)$ are as follows. In Figure 1, the pairs which cross the seam are $(3, -1)$ and $(4, 0)$, giving $w(D) = 2$. Figure 2 has $w(D) = 0$. Figure 3 has $w(D) = 1$, corresponding to the pair $(1, 0)$. A moment's thought shows that $w(u^n) = n$ for any $n \in \mathbb{Z}$, which motivates the definition.

We recall the definition of an annular involution of the symmetric group from [**2**, Lemma 6.2].

Definition 3.2.3. *An involution $S \in \mathcal{S}_n$ is annular if and only if, for each pair i, j interchanged by S $(i < j)$, we have*
(a) $S[i, j] = [i, j]$ *and*
(b) $[i, j] \cap \mathrm{Fix}S = \emptyset$ *or* $\mathrm{Fix}S \subseteq [i, j]$.

We write $S \in I(t)$ if S has t fixed points, and we write $S \in \mathrm{Ann}(\mathbf{n})$ if S is annular.

Lemma 3.2.4. *The diagrams of D_n with vertical edges are in canonical bijection with triples $[S_1, S_2, w]$ where $S_1, S_2 \in \mathrm{Ann}(\mathbf{n}) \cap I(t)$ for $t > 0$, and $w \in \mathbb{Z}$.*

Proof. The correspondence is given as follows. Consider a diagram D. The involution S_1 exchanges points i and j if and only if i is connected to j in the top circle of D. Similarly S_2 exchanges points i and j if and only if i is connected to j in the bottom circle of D. One can easily see that S_1 and S_2 are annular. Finally $w = w(D)$; this determines which points are connected by the vertical edges (it is enough to specify one vertical edge, by the non-intersection criterion).

Conversely consider a triple $[S_1, S_2, w]$ as required. The hypothesis $t > 0$ guarantees that each of S_1 and S_2 each has a fixed point. If $S \in S_1, S_2$ exchanges points i and j, it now follows from part (b) of the definition that we can determine which way round the cylinder the corresponding edge in the diagram goes. Using part (a) we can show that no intersections occur, so S_1 determines the horizontal edges in the top of the diagram and S_2 determines those in the bottom. As regards w, observe that one can disconnect all the vertical edges from a diagram and reconnect the top end of each one to the next available point to the right. This increases $w(D)$ by 1. Conversely one can decrease it by one. Thus w determines the endpoints of all the vertical edges, which completes the proof. $\square$

Definition 3.2.5. *For $S_1, S_2 \in \mathrm{Ann}(\mathbf{n}) \cap I(t)$ where $t > 0$, we define $r(S_1, S_2)$ to be the smallest nonnegative integer (which will be* 0 *or* 1*, following Proposition 2.3.4) satisfying $[S_1, S_2, r(S_1, S_2)] \in O_n$.*

3.3 q-Jones Algebras in Odd Rank.

In §3.3, we assume that n is an odd number. (We deal with this before the even case because the even case is already better understood.)

We now put an equivalence relation, $\sim$, on the basis elements $[S_1, S_2, w]$.

Definition 3.3.1. *We say that $[S_1, S_2, w] \sim [T_1, T_2, w']$ if and only if $S_1 = T_1$, $S_2 = T_2$ and w and w' are congruent modulo t, where $S_1, S_2 \in \mathrm{Ann}(\mathbf{n}) \cap I(t)$.*

We say a basis element $[S_1, S_2, w]$ is in root position if and only if $0 \leq w < t$. Clearly the set of basis elements in root position is a set of equivalence class representatives for $\sim$. We usually denote such an element by $[S_1, S_2, \tau_i]$ to emphasize that it is in root position.

Lemma 3.3.2. *The subspace spanned by elements of the form*

$$\{[S_1, S_2, w] - q^s[T_1, T_2, w'] : [S_1, S_2, w] \sim [T_1, T_2, w'] \text{ and } \ w = w' + ts\},$$

where $S_1 \in \mathrm{Ann}(\mathbf{n}) \cap I(t)$, is an ideal of D_n. We denote it by $\omega(q)$.

Proof. It is clear from the diagram calculus that the product of any two basis elements of D_n is a multiple of another one. If

$$[X_1, X_2, w][Y_1, Y_2, w'] = \beta[Z_1, Z_2, w'']$$

for some scalar β then one can also easily see that

$$[X_1, X_2, w][Y_1, Y_2, w' + t] = \beta[Z_1, Z_2, w'' + t].$$

The claim now follows. □

Definition 3.3.3. *The q-Jones algebra over $K(q)$ (where q is an indeterminate) is the quotient of D_n by $\omega(q)$.*

Remark 3.3.4. When $q = 1$, one recovers the usual Jones algebra (see [**2**, Proposition 6.14]).

Lemma 3.3.5. *The q-Jones algebra has a basis given by the images of the elements*

$$\{[S_1, S_2, \tau_i] : S_1, S_2 \in \mathrm{Ann}(\mathbf{n}) \cap I(t), \tau_i \in \{0, 1, \ldots, t-1\}\}$$

Proof. It is clear that the given elements span the algebra.

Suppose for a contradiction that there is some nontrivial linear relation between the elements. Then we may assume that the coefficients of this relation lie in $K[q]$ and that not all of them have $(q-1)$ as a factor. Putting $q = 1$ and using Remark 3.3.4, we now obtain a nontrivial linear relation for the corresponding basis elements of the Jones algebra, which is a contradiction. □

3.4 Representation theory of the q-Jones algebra in odd rank.

We can now prove that the q-Jones algebra is cellular. The motivation behind doing this is that we will then have bases for all its finite dimensional irreducible modules over K. The methods are the same as those in [**2**, §6].

We specialize the indeterminate q to a fixed, nonzero element of K. We also define the set $\mathcal{T}(n)$ to be the set of odd natural numbers up to and including n. We abuse notation by identifying an element $[S_1, S_2, w]$ where $0 \leq w < t$ with its image in $J_q(n)$.

Definition 3.4.1. *Let w_0 be the longest element of $\mathcal{S}_n$. If $S \in \mathrm{Ann}(\mathbf{n})$, we define $S^* \in \mathrm{Ann}(\mathbf{n})$ to be the element $w_0^{-1} S w_0$.*

Proposition 3.4.2. *Let $t \in \mathcal{T}(n)$. We factorise*

$$x^t - q = \prod_{i=1}^{t} (x - r_i(t, q))$$

and write

$$f_j(x, q) = \prod_{i>j} (x - r_i(t, q)) = \sum_{i=0}^{t-j} r_{ij}(t, q) x^i$$

for $j = 1, \ldots, t$. Write $f_0(x, q) = 1$ so that $r_{00}(0) = 1$. Then a cell datum for $J_q(n)$ over K is given by $(\Lambda, M, C, *)$ defined as follows:

(a) *$\Lambda = \{(t, j) : t \in \mathcal{T}(n), j \in \mathbf{t}\}$, ordered lexicographically, i.e. $(t', j') < (t, j)$ if and only if $t' < t$ or $t = t', j' < j$.*
(b) *For $(t, j) \in \Lambda$, $M(t, j) = \mathrm{Ann}(\mathbf{n}) \cap I(t)$.*
(c) *If $S_1, S_2 \in M(t, j)$ then $C^{t,j}_{S_1,S_2} = \sum_{i=0}^{t-j} r_{ij}(t, q)[S_1, S_2^*, i]$.*
(d) *The anti-automorphism, $*$, is defined by $[S_1, S_2, w]^* = [S_2^*, S_1^*, w]$.*

Proof. We check that the three conditions of Definition 3.1.1 hold. Condition 1 comes from Lemma 3.3.5.

To prove condition 2, we observe that applying $*$ to a basis element $[S_1, S_2^*, w]$ in root position gives another element in root position. The fact that the map is an anti-automorphism is clear from the diagrammatic viewpoint—the cylinders are being rotated so they are upside-down.

The proof of condition 3 is the same as [**2**, Theorem 6.15], with a few minor and obvious changes. □

The motivation behind proving Proposition 3.4.2 is that the theory of cellular algebras allows one to construct a complete list of finite-dimensional irreducible modules for $J_q(n)$.

Definition 3.4.3. *Let $\lambda = (t, j) \in \Lambda$. The module $W(t, j, q)$ for $J_q(n)$ has a basis given by $\{C_S : S \in M(t, j)\}$ with action specified as follows. If $a \in J_q(n)$ and*

$$a.C^{t,j}_{S,T} \equiv \sum_{S' \in M(t,j)} c_a(S', S) C^{t,j}_{S',T} \mod J_q(n)\ (< \lambda)$$

for any suitable T (for example, S), then the left action of a on $W(t, j, q)$ is defined by

$$a.C_S = \sum_{S' \in M(t,j)} c_a(S', S) C_{S'}.$$

Remark 3.4.4. The theory of cellular algebras in [**2**] ensures that this is well-defined.

Proposition 3.4.5. *The modules $W(t, j, q)$, as $r_j(t, q)$ ranges over the t-th roots of q, are a complete set of irreducible representations for $J_q(n)$ over K.*

Proof. This is essentially a restatement of [**2**, Corollary 6.17], after making the necessary trivial changes to [**2**, Scholium 6.16]. Note that the parameter δ occurring in [**2**, Corollary 6.17] is equal to $[2] = v + v^{-1} \neq 0$ in our case. □

Proposition 3.4.6. *Let ψ be a representation corresponding to the irreducible module $W(t, j, q)$, and let $[S_1, S_2, w]$ be a basis element of $J_q(n)$ arising from the set $M(t, j)$. Then*
$$\psi([S_1, S_2, w + 1]) = r_j(t, q).\psi([S_1, S_2, w]).$$

Proof. This is a corollary of the proof of Proposition 3.4.2. For more details see [**2**, (6.15.1), (6.15.2)]. □

Remark 3.4.7. Proposition 3.4.6 gives a convenient way to test whether the representations $W(t, j, q)$ and $W(t, j', q)$ are distinct. We will use this fact later in the classification of the simple modules for $TL(\widehat{A}_{n-1})$.

3.5 q-Jones Algebras in Even Rank.

We now highlight the differences between the odd rank situation and the corresponding even rank situation. We treat the case of even rank in less detail than the case of odd rank studied in §3.3 and §3.4, because the situation is already better understood and the techniques are very similar.

In §3.5, n is even and the algebra O_n takes over the role which D_n played in §3.3 and §3.4. Since n is even, any diagram has an even number of vertical edges, and so the relation $\sim$ of Definition 3.3.1 carries over naturally to an equivalence relation on the basis elements of O_n.

For n even, we set $\mathcal{T}(n) := \{0, 2, 4, \dots, n\}$.

Lemma 3.5.1. *The diagrams of O_n with vertical edges are in canonical bijection with triples $[S_1, S_2, r(S_1, S_2) + w]$ where $t \in \mathcal{T}(n)\backslash\{0\}$, $S_1, S_2 \in \mathrm{Ann}(\mathbf{n}) \cap I(t)$, and $w \in 2\mathbb{Z}$.*

Proof. This is along the same lines as the proof of Lemma 3.2.4. □

Remark 3.5.2. The hypothesis that there exists a vertical edge in the diagram is necessary, otherwise one could form a counterexample using any diagram with no vertical edges!

Definition 3.5.3. *The R-module $\omega'(q)$ of O_n is the span of elements of O_n of the form given in Lemma 3.3.2 where $[S_1, S_2, w]$, and hence $[T_1, T_2, w']$, involve vertical edges. The ideal I_0 of O_n is the span of the elements of O_n with no vertical edges.*

The q-Jones algebra over $K(q)$ (where q is an indeterminate) is the quotient of O_n by the ideal $\omega'(q) + I_0(q)$.

Remark 3.5.4. It is clear from the diagrammatic viewpoint that $I_0(q)$ is an ideal. One proves that $\omega'(q) + I_0(q)$ is an ideal by using an argument identical to that of the proof of Lemma 3.3.2.

It should be noted that the term "q-Jones algebra" is misleading in even rank, because specializing q to 1 gives a smaller algebra than Jones' annular algebra. In fact, we obtain the quotient of the Jones algebra by its ideal $J_0(n)$ spanned by basis elements with no through-strings. The reason for factoring out I_0 is that irreducible modules M satisfying $I_0.M = M$ behave rather differently from those satisfying $I_0.M = 0$. In this paper we are interested in the latter class of modules.

Lemma 3.5.5. *The q-Jones algebra $J_q(n)$, for even n, has a basis given by the images of the elements*

$$\{[S_1, S_2, r(S_1, S_2) + 2w]\},$$

where $t \in \mathcal{T}(n)\backslash\{0\}$, $S_1, S_2 \in \mathrm{Ann}(\mathbf{n}) \cap I(t)$ and $w \in \{0, 1, \ldots, t/2 - 1\}$.

Proof. As in Lemma 3.3.5, it is clear that the given elements span the algebra.

Suppose for a contradiction that there is some nontrivial linear relation between the elements. Then we may assume that the coefficients of this relation lie in $K[q]$ and that not all of them have $(q - 1)$ as a factor. Putting $q = 1$, we now obtain a nontrivial linear relation for the corresponding basis elements of $J(n)/J_0(n)$, and thus a nontrivial linear relation in $J(n)$, which is a contradiction. □

We will abuse notation again by writing $[S_1, S_2, w]$ (one of the elements appearing in Lemma 3.5.5) for its image in $J_q(n)$.

We now give a cell datum for $J_q(n)$ in even rank.

Proposition 3.5.6. *Define $\mathcal{T}'(n) := \{1, 2, \ldots, n/2\}$, and let $t \in \mathcal{T}'(n)$. We factorise*

$$x^t - q = \prod_{i=1}^{t} (x - r_i(t, q))$$

and write

$$f_j(x, q) = \prod_{i>j} (x - r_i(t, q)) = \sum_{i=0}^{t-j} r_{ij}(t, q) x^i$$

for $j = 1, \ldots, t$. Write $f_0(x, q) = 1$ so that $r_{00}(0) = 1$. Then a cell datum for $J_q(n)$ over K is given by $(\Lambda, M, C, *)$ defined as follows:

(a) *$\Lambda = \{(t, j) : t \in \mathcal{T}'(n), j \in \mathbf{t}\}$, ordered lexicographically, i.e. $(t', j') < (t, j)$ if and only if $t' < t$ or $t = t', j' < j$.*
(b) *For $(t, j) \in \Lambda$, $M(t, j) = \mathrm{Ann}(\mathbf{n}) \cap I(2t)$.*
(c) *If $S_1, S_2 \in M(t, j)$ then $C^{t,j}_{S_1,S_2} = \sum_{i=0}^{t-j} r_{ij}(t, q)[S_1, S_2^*, r(S_1, S_2^*) + 2i]$.*
(d) *The anti-automorphism, $*$, is defined by $[S_1, S_2, w]^* = [S_2^*, S_1^*, w]$.*

Proof. This uses exactly the same techniques as Proposition 3.4.2. □

Remark 3.5.7. Using this cell datum, one can state and prove results exactly analogous to Definition 3.4.3, Remark 3.4.4, Proposition 3.4.5, Proposition 3.4.6 and Remark 3.4.7. The only appreciable differences are that the role of $\mathcal{T}(n)$ in the odd rank is played by $\mathcal{T}'(n)$ in the even rank, and that the role of t in the odd rank is played by $t/2$ in the even rank.

4. Classification of Simple Modules for $TL(\widehat{A}_{n-1})$

To classify the simple modules for $TL(\widehat{A}_{n-1})$, we find once again that it is helpful to consider the cases of odd and even n separately. We tackle the case of odd rank first, because it is less well understood.

4.1 Simple modules in odd rank.

The technique in odd rank is to calculate the simple modules for D_n and study their behaviour upon restriction to O_n and then to $TL(\widehat{A}_{n-1})$. We also recall the automorphism ε from §2.

Proposition 4.1.1. *The irreducible representations ψ for D_n are parametrised by pairs (α_t, t), where $t \in \mathcal{T}(n)$ and $\alpha_t \in K \backslash 0$. The scalar α_t is that appearing in the equation*
$$\psi([S_1, S_2, w+1]) = \alpha_t.\psi([S_1, S_2, w]),$$
where $S_1, S_2 \in \mathrm{Ann}(\mathbf{n}) \cap I(t)$.

Proof. Let M be a simple module for D_n. Let I_t be the ideal of D_n spanned by all diagrams with at most t vertical edges. Then there exists a unique (odd) number t such that $I_t.M = M$ but $I_{t-2}.M = 0$, interpreting $I_{-1} = 0$. We also see that M is an irreducible module for D_n/I_{t-2}. Clearly, $t \in \mathcal{T}(n)$.

One sees from the diagrammatic viewpoint that u^n lies in the centre of D_n. By Schur's Lemma, u^n acts as a scalar, α, on M. The ideal generated by $u^n - \alpha$ is the same as the ideal $\omega(q)$ in Definition 3.3.3, for $q = \alpha$, so M is a simple module for the q-Jones algebra $J_q(n)$ for this value of q. Proposition 3.4.5 now shows that the irreducibles are parametrised by the set of t-th roots of α. (Note that we allow the possibility that α does not have t distinct t-th roots and there are fewer than t irreducibles.) The fact that nonisomorphic irreducibles have different parameters comes from Proposition 3.4.6 and Remark 3.4.7. Conversely, any nonzero $\alpha \in K$ turns up as one of the numbers $r_j(t, q)$ for $q = \alpha^t$.

This means that all the irreducibles for D_n are parametrised by pairs (α_t, t), where t is as above and $\alpha_t \in K \backslash 0$. The last assertion follows from Proposition 3.4.6. □

Proposition 4.1.2. *The irreducible representations ρ for O_n are parametrised by pairs (α_t, t), where $t \in \mathcal{T}(n)$ and $\alpha_t \in K \backslash 0$. The scalar α_t is that appearing in the equation*
$$\rho([S_1, S_2, w+2]) = \alpha_t.\rho([S_1, S_2, w]),$$
where $S_1, S_2 \in \mathrm{Ann}(\mathbf{n}) \cap I(t)$.

Proof. Let L be a simple module for D_n, and let L' be a simple O_n-submodule of L (using the O_n-module structure suggested by Lemma 2.3.5). Since u^n acts as a scalar on L, it acts as a scalar on L', and in particular $u^n L' = L'$. However, $u^2 L' = L'$ since $u^2 \in O_n$. Since n is odd and u is invertible, this implies $u.L' = L'$, and thus $L = L'$. Furthermore, L is determined by L' and the scalar by which u^n acts.

It is clear that two irreducible modules L_1 and L_2 for D_n restrict to nonisomorphic modules for O_n unless their parameters (α_1, t_1) and (α_2, t_2) satisfy $t_1 = t_2$ and $\alpha_1 = \pm\alpha_2$. (This is because
$$\psi([S_1, S_2, w+2]) = \alpha^2.\psi([S_1, S_2, w]),$$
where ψ is a representation of D_n corresponding to a pair (α, t) and $S_1, S_2 \in \mathrm{Ann}(\mathbf{n}) \cap I(t)$.) If $\mathrm{char}K = 2$ then necessarily $L_1 = L_2$. Otherwise, let ψ_1 be a representation affording L_1. We claim that if L_1 is given by the pair (α_t, t), then $\psi_1.\varepsilon$ is an irreducible representation for D_n corresponding to the pair $(-\alpha_t, t)$ and thus affording $L_2 \neq L_1$. This is because
$$\psi([S_1, S_2, w+1]) = \alpha.\psi([S_1, S_2, w]) \Rightarrow \psi(\varepsilon([S_1, S_2, w+1])) = -\alpha.\psi(\varepsilon([S_1, S_2, w])).$$
Irreducibility follows from the fact that L_1 and L_2 have the same dimension. Thus $\psi_1.\varepsilon$ affords L_2, although ψ_1 and $\psi_1.\varepsilon$ give the same representation of O_n, since ε fixes O_n.

The result now follows from these observations and Frobenius reciprocity. □

Theorem 4.1.3. *The nontrivial irreducible representations ρ for $TL(\widehat{A}_{n-1})$ are parametrised by pairs (α_t, t), where $t \in \mathcal{T}(n)\backslash\{n\}$ and $\alpha_t \in K\backslash 0$. The scalar α_t is that appearing in the equation*

$$\rho([S_1, S_2, w+2]) = \alpha_t.\rho([S_1, S_2, w]),$$

where $S_1, S_2 \in \mathrm{Ann}(\mathbf{n}) \cap I(t)$. The only other irreducible representation is the "trivial" representation of dimension 1 which sends the identity to 1 and all the generators E_i to 0.

Proof. Recall that $TL(\widehat{A}_{n-1})$ and O_n are filtered by two-sided ideals indexed by the number of vertical edges in the diagrams. Apart from the top section, which contains only the identity in $TL(\widehat{A}_{n-1})$ but is infinite in O_n, the sections of this filtration by ideals are the same (in fact, the ideals themselves are the same). It follows that the simple modules are the same, except for those corresponding to the top section. In the latter case, only the "trivial" module (as described in the statement of the Theorem) for $TL(\widehat{A}_{n-1})$ survives. □

4.2 Simple modules in even rank.

Similar techniques can also be used to classify the simple modules for $TL(\widehat{A}_{n-1})$ where n is even. As usual we exclude the case where $I_0.M = M$ and I_0 is as in Definition 3.5.3.

Proposition 4.2.1. *The irreducible representations ρ for O_n satisfying $\rho(I_0) = 0$ are parametrised by pairs (α_t, t), where $t \in \mathcal{T}'(n)$ and $\alpha_t \in K\backslash 0$. The scalar α_t is that appearing in the equation*

$$\rho([S_1, S_2, w+2]) = \alpha_t.\rho([S_1, S_2, w]),$$

where $S_1, S_2 \in \mathrm{Ann}(\mathbf{n}) \cap I(2t)$.

Proof. Let M be the simple module corresponding to ρ. Again, u^n acts as a scalar, α, on M, by Schur's lemma. Thus the ideal generated by $u^n - \alpha$ factors through M, and we obtain an irreducible representation for $J_q(n)$ with $q = \alpha$. The list of such representations is indexed by the t-th roots of α, and they are distinguishable as in the statement of the proposition by the usual techniques. As in Proposition 4.1.1, such a representation exists for any α_t, which completes the proof. □

Finally we rederive the result in [**4**] which parametrises the simple modules for $TL(\widehat{A}_{n-1})$ in even rank.

Theorem 4.2.2. *The nontrivial irreducible representations ρ for $TL(\widehat{A}_{n-1})$ which satisfy $\rho(I_0) = 0$ are parametrised by pairs (α_t, t), where $t \in \mathcal{T}'(n)\backslash\{n/2\}$ and $\alpha_t \in K\backslash 0$. The scalar α_t is that appearing in the equation*

$$\rho([S_1, S_2, w+2]) = \alpha_t.\rho([S_1, S_2, w]),$$

where $S_1, S_2 \in \mathrm{Ann}(\mathbf{n}) \cap I(2t)$. The only other irreducible representation is the "trivial" representation of dimension 1 which sends the identity to 1 and all the generators E_i to 0.

Proof. This follows the same lines as Theorem 4.1.3, except that it uses Proposition 4.2.1 instead of Proposition 4.1.2. □

Acknowledgements

The author would like to thank C.K. Fan for some helpful discussions. The author is also grateful to K. Erdmann and the referee for numerous helpful comments on the manuscript.

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