HYPEROCTAHEDRAL SCHUR ALGEBRAS

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Dedicated to Professor J.A. Green on his 70th birthday

Abstract. We study the centralising algebra of a natural action of the hyperoctahedral group (i.e. a finite Weyl group of type B_r) on the r-th tensor power of a $2n$-dimensional space. The centralising algebra of this is shown to have a product rule similar to Schur's product rule in type A. We deform this action to an action of the Hecke algebra of type B and study the associated centralising algebra of type B and its dual. We introduce and study q-permutation modules for the algebra.

0. Introduction

The classical Schur algebra, $S(n, r)$, is a finite-dimensional associative algebra which has a basis consisting of certain elements $\xi_{i,j}$, as described in [**7**, §2.3]. The product of two such elements can be computed using Schur's Product Rule ([**7**, 2.3b]). This is an explicit but elegant formula based on the combinatorics of certain symmetric group actions. A natural question to ask is: what happens if we replace the symmetric group in the definition of Schur's Product Rule by another group? In Proposition 2.2.7, we investigate this in the case where the group is a hyperoctahedral group, i.e. a Weyl group of type B, with a certain natural action reminiscent of the action of the symmetric group. We call this object the hyperoctahedral Schur algebra.

In §3, the situation is generalised from Weyl groups to Hecke algebras. We thus find an analogue of the q-Schur algebra in type B, which was first introduced by Geck and Hiss in [**5**, pp. 173–227]. We show in §4 that our algebra is Morita equivalent to an algebra appearing in the work of Gruber and Hiss ([**6**] §7.2]), and to the original algebra of [**5**].

It should be remarked that our algebra arises as the q-analogue of a combinatoric object, but the algebra of [**6**] arises as an endomorphism ring of a certain sum of "q-permutation modules" for the Hecke algebra of type B. The hyperoctahedral Schur algebra of the title is not discussed in [**6**], although it can of course be recovered by setting the parameters in the q-Schur algebra (of type B) to 1.

We also show that the q-Schur algebra of type B is related to an algebra appearing in [**4**]. The basis we give for our algebra is naturally compatible with the bases

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in [**4**]. This leads to an IC ("intersection cohomology") basis for our algebra, in the case where the parameters are equal.

Finally we show in §4.5 that, subject to certain restrictions on the algebra (similar to the restrictions required in type A), we have a double centraliser property involving the q-Schur algebra of type B and the Hecke algebra of type B.

1. Finite Weyl groups of type B

1.1 Basic properties of hyperoctahedral groups.

The hyperoctahedral groups which we use in this paper are the finite Weyl groups $W = W(B_r)$ of type B_r. These are given (as in [**8**, §1]) by generators $s_1, s_2, \ldots, s_r$ and defining relations

$$\begin{aligned} s_i^2 &= 1, \\ s_i s_j &= s_j s_i \quad \text{if } |i-j| > 1, \\ s_i s_j s_i &= s_j s_i s_j \quad \text{if } |i-j| = 1 \text{ and } i, j < r, \\ s_{r-1} s_r s_{r-1} s_r &= s_r s_{r-1} s_r s_{r-1}. \end{aligned}$$

(The last three relations are known as *braid relations*.)

We see from [**8**, §2] that $W(B_r)$ is isomorphic to the wreath product $\mathbb{Z}_2 \wr \mathcal{S}_r$. Once this isomorphism is fixed, we see that W acts naturally and faithfully on the right on the ordered set of symbols $\{1, 2, \ldots, r, \bar{r}, \overline{r-1}, \ldots, \bar{1}\}$. We use the representation of W implicit in [**8**, §2], where the generators s_i act as follows. If $i < r$, then s_i exchanges i with $i+1$ and exchanges $\bar{i}$ with $\overline{i+1}$, leaving the other symbols fixed. The generator s_r exchanges the symbols r and $\bar{r}$, leaving the others fixed. We thus have the following well-known lemma:

Lemma 1.1.1. *The group $W(B_r)$ embeds in the group $W(A_{2r-1})$ as follows. The generator s_i of $W(B_r)$ (for $i < r$) is sent to the length two element $s'_i s'_{2r-i}$ of $W(A_{2r-1})$ which is identified with the double transposition $(i, i+1)(2r-i, 2r-i+1)$. The generator s_r of $W(B_r)$ is sent to the simple transposition s'_r of $W(A_{2r-1})$ which is identified with the simple transposition $(r, r+1)$.*

Proof. This is clear from the preceding discussion. □

We now define a set similar to the set $I(n,r)$ which is used in the theory of ordinary Schur algebras.

Definition 1.1.2. *We define the set $I_B(2n,r)$ to be the set of r-tuples*

$$(i_1, i_2, \dots, i_r)$$

from the alphabet $\{1,2,\dots,n,\bar{n},\overline{n-1},\dots,\bar{1}\}$. We call the elements $\{1,2,\dots,n\}$ unbarred elements, and we call the elements $\{\bar{n},\overline{n-1},\dots,\bar{1}\}$ barred elements. We write $|i| = i$ if i is unbarred, and $|\bar{i}| = i$ if $\bar{i}$ is barred.

Let V_{2n} be a vector space of dimension $2n$ with ordered basis

$$\{e_1, e_2, \dots, e_n, e_{\bar{n}}, e_{\overline{n-1}}, \dots, e_{\bar{1}}\}.$$

By analogy with the situation in type A, one would like to be able to define a natural action of $W(B_r)$ on $I_B(2n,r)$, and hence on the tensor space $V^{\otimes r}$, which resembles the action of W on the symbols $\{1,\dots,n,\bar{n},\dots,\bar{1}\}$.

A natural question now presents itself: is there an analogue of Schur–Weyl duality involving the group $W(B_r)$ and its centralising algebra with respect to the module $V_{2n}^{\otimes r}$? One would hope that the centralising algebra would be an object similar to the ordinary Schur algebra which was presented in [**7**]. One may also ask whether there exists a hyperoctahedral version of the q-Schur algebra (introduced in [**2**]). We address this question later in the paper.

1.2 Permutation notation and length criteria.

First, we give a permutation representation for an element of the Weyl group $W = W(B_r)$. This is well known.

Lemma 1.2.1. *The elements $w \in W$ are naturally parametrised by the elements of $I_B(2r,r)$ of the form $(i_1,\dots,i_r)$, where $\{|i_m| : 1 \le m \le r\} = \{1,\dots,r\}$. The correspondence is given by the criterion that $i_m = (m)w$.*

Proof. It is immediate from the realisation of W as a wreath product that $(\bar{m})w = \overline{(m)w}$ for all w. Thus w is determined by the images $(m)w$ for $m \in \{1,\dots,r\}$.

We now need to show that any element of the given form turns up as the image of an element of W. Since the size of the parametrising set is $2^r.r!$, this follows by an easy counting argument. $\square$

Note. Note that this notation does not correspond to the disjoint cycle notation for elements of the symmetric group, despite superficial similarities.

The length $\ell(w)$ of an element $w \in W$ is the length of the shortest word in the Coxeter generators s_i which is equal to it. We now present a length criterion for elements of W in terms of the permutation representation described above.

Lemma 1.2.2. *Let* **a** *be an element of* $I_B(2r, r)$ *representing an element of* w. *Clearly* $\ell(ws_i) = \ell(w) \pm 1$. *If* $i < r$ *then* $\ell(ws_i) = \ell(w) + 1$ *if and only if* $a_i < a_{i+1}$. *Furthermore,* $\ell(ws_r) = \ell(w) + 1$ *if and only if* a_r *is unbarred.*

Proof. This follows from the corresponding well-known length identities in type A. Let us write $\iota(w)$ for the element of $W(A_{2r-1})$ corresponding to w, as in Lemma 1.1.1.

If $i < r$, then we see that

$$\ell(\iota(w)\iota(s_i)) = \ell(\iota(w)) \pm 2,$$

where the $\pm$ corresponds to the $\pm$ in the statement of the lemma. This means that

$$\ell(\iota(w)s_i) = \ell(\iota(w)) \pm 1.$$

We obtain a plus if $(i)\iota(w) < (i + 1)\iota(w)$, and a minus otherwise, as required. The same applies to w because of the way $W(B_r)$ is embedded in the symmetric group on the letters $\{1, \dots, n, \bar{n}, \dots, \bar{1}\}$. (This is essentially the observation made in the first paragraph of the proof of Lemma 1.2.1.) Using the fact that $a_i = (i)w$ we complete the proof of the case $i < r$.

If $i = r$, we find that

$$\ell(\iota(w)s_i) = \ell(\iota(w)) \pm 1,$$

because $\iota(s_r) = s_r$. The plus occurs if and only if $(r).\iota(w) < (\bar{r}).\iota(w)$, which is equivalent to saying that $(r)\iota(w)$ is unbarred. This happens if and only if $(r)w$ is unbarred, i.e. a_r is unbarred, as required. $\square$

We now give a criterion for identifying the shortest right coset representatives with respect to certain parabolic subgroups of W.

Lemma 1.2.3. *Let P be a parabolic subgroup of W which does not contain the generator s_r. Consider an interval $1 \leq a < b \leq r$ such that $(m, m+1) \in P$ for each $a \leq m < b$, but where a is minimal with this property and b is maximal with this property.*

Then $d \in W$ is a distinguished right coset representative (i.e. one of minimal length) of P in W if for all pairs (a, b) as above, there exists c with $a \leq c \leq b$ such that the representation for d (in the sense of Lemma 1.2.1) has the property that the numbers $a, a+1, \ldots, c$ appear unbarred in ascending order from left to right, and the numbers $\bar{b}, \overline{b-1}, \ldots, \overline{c+1}$ appear barred in the order given.

Proof. We prove this by embedding $W(B_r)$ in $W(A_{2r-1})$. The subgroup P corresponds to a parabolic subgroup of $W(A_{2r-1})$ in an obvious way which preserves the notion of distinguished coset representatives. The group of type A has a well-known criterion for testing for distinguished coset representatives, and our criterion follows easily from it. $\square$

2. Tensor space

2.1 Action of $W(B_r)$ on tensor space.

In order to give an action of $W = W(B_r)$ on tensor space, we first introduce a permutation representation for W, as follows.

Lemma 2.1.1. *We have a natural action of W on $I_B(2n, r)$, defined by the condition that $\mathbf{i}.s$ is such that*

$$(i_1, \ldots, i_r).s = \begin{cases} (i_{(1)s}, \ldots, i_{(r)s}) & \text{if } s \neq s_r, \\ (i_1, \ldots, i_{r-1}, \overline{i_r}) & \text{if } s = s_r, \end{cases}$$

where we interpret $\bar{\bar{i}}$ to mean i and where s is a generating involution. This gives rise to an action on $I_B(2n, r) \times I_B(2n, r)$, given by $(\mathbf{i}, \mathbf{j}).\pi = (\mathbf{i}.\pi, \mathbf{j}.\pi)$. We write $a \sim b$ to mean that a and b are in the same W-orbit, where a and b are both elements of $I_B(2n, r)$ or both elements of $I_B(2n, r) \times I_B(2n, r)$.

Proof. It is an easy matter to check that the defining relations of $W(B_r)$ are satisfied. $\square$

Next, we transfer this action of W to the tensor power $V^{\otimes r} = V_{2n}^{\otimes r}$, where V is a $\mathbb{Q}$-vector space. (The symbols n and r will be reserved with this meaning.)

Lemma 2.1.2. *There is a right action of W on $V^{\otimes r}$ in which the generator $s = s_a$ acts on the natural basis vectors as follows:*

$$(e_{i_1} \otimes \cdots \otimes e_{i_r}).s_a = \begin{cases} e_{i_{(1)s}} \otimes \cdots \otimes e_{i_{(r)s}} & \text{if } s \neq s_r, \\ e_{i_1} \otimes \cdots \otimes e_{i_{r-1}} \otimes e_{\overline{i_r}} & \text{if } s = s_r, \end{cases}$$

where we interpret $\bar{\bar{i}}$ to mean i.

Proof. This is simply the permutation module corresponding to the permutation representation of Lemma 2.1.1. $\square$

We can relate this action to the results in §1.2.

Lemma 2.1.3. *Let* **a** *represent an element $w \in W$. Suppose $n \geq r$. Then*

$$(e_1 \otimes e_2 \otimes \cdots \otimes e_r).w = e_{a_1} \otimes \cdots \otimes e_{a_r}.$$

Proof. It is enough to show that

$$(e_1 \otimes e_2 \otimes \cdots \otimes e_r).w = e_{(1)w} \otimes \cdots \otimes e_{(r)w}.$$

This is proven by an easy induction on $\ell(w)$, using the definition of the action in Lemma 2.1.2. $\square$

2.2 The centralising algebra of $W(B_r)$.

We now consider the centralising algebra of the action of W on tensor space as in §2.1.

Definition 2.2.1. *Let $\eta \in \text{End}(V^{\otimes r})$. For $\mathbf{i}, \mathbf{j} \in I_B(2n, r)$, we define the coordinate functions $c_{\mathbf{i},\mathbf{j}}$ via*

$$\eta(e_{\mathbf{j}}) = \sum_{\mathbf{i} \in I_B(2n,r)} c_{\mathbf{i},\mathbf{j}} e_{\mathbf{i}}.$$

Here, $e_{\mathbf{i}}$ denotes the tensor $e_{i_1} \otimes \cdots \otimes e_{i_r}$ where $\mathbf{i} = (i_1, \ldots, i_r)$.

Definition 2.2.2. *We denote the centralising algebra of the right action of W on $V_{2n}^{\otimes r}$ by $H(2n,r)$.*

In order for an element $\eta \in \mathrm{End}(V^{\otimes r})$ to lie in $H(2n,r)$, the coordinate functions are required to satisfy certain relations. We now determine these relations.

Lemma 2.2.3. *The coordinate functions on $H(2n,r)$ satisfy the relations*

$$c_{\mathbf{i},\mathbf{j}} = c_{\mathbf{i}.s_a,\mathbf{j}.s_a}$$

for all generating involutions s_a. These relations are necessary and sufficient.

Proof. To say that $\eta \in H(2n,r)$ is equivalent to saying that its action commutes with the action of all the s_a.

This is essentially known for the case $a < r$, because such conditions are known to be necessary and sufficient to ensure that elements of $\mathrm{End}(V^{\otimes r})$ commute with the action of the symmetric group by place permutation of the tensors. (This fact is known and obvious from the situation in type A.) So we only need to check that each element $h \in H(2n,r)$ commutes with the action of s_r.

Comparing the action of h on two basis vectors $e_{\mathbf{i}}$ and $e_{\mathbf{i}.s_r}$ and using the facts that $e_{\mathbf{i}.s_r} = e_{\mathbf{i}}.s_r$ (by definition) and that the left action of h commutes with the right action of s_r, we obtain the proof of the statement for $a = r$. $\square$

Corollary 2.2.4. *Let Ω be a transversal for the action of W on $I_B(2n,r) \times I_B(2n,r)$. For any $w \in W$, we have $c_{\mathbf{i},\mathbf{j}} = c_{\mathbf{i}.w,\mathbf{j}.w}$, and the set $\{c_{\mathbf{i},\mathbf{j}} : (\mathbf{i},\mathbf{j}) \in \Omega\}$ is a basis for the coordinate ring $H^*(2n,r)$ of $H(2n,r)$.*

Proof. The fact that $c_{\mathbf{i},\mathbf{j}} = c_{\mathbf{i}.w,\mathbf{j}.w}$ follows by induction on $\ell(w)$ and Lemma 2.2.3. The second assertion now follows. $\square$

Definition 2.2.5. *We define the element $\eta_{\mathbf{ij}} \in \mathrm{End}(V^{\otimes r})$ by the conditions*

$$c_{\mathbf{ab}}(\eta_{\mathbf{ij}}) = \begin{cases} 1 & \text{if } (\mathbf{a},\mathbf{b}) \sim (\mathbf{i},\mathbf{j}), \\ 0 & \text{otherwise.} \end{cases}$$

Note that these elements $\eta_{\mathbf{ij}}$ satisfy the condition $c_{\mathbf{ab}}(\eta_{\mathbf{ij}}) = c_{\mathbf{a}.w,\mathbf{b}.w}(\eta_{\mathbf{ij}})$ for any $w \in W$ and are therefore, by Corollary 2.2.4, elements of $H(2n,r)$.

Proposition 2.2.6. *The set* $\{\eta_{\mathbf{ij}} : (\mathbf{i}, \mathbf{j}) \in \Omega\}$ *is a basis for the algebra* $H(2n, r)$.

Proof. This follows from standard properties of permutation modules. $\square$

The following proposition is the analogue of Schur's product rule ([**7**, 2.3b]) for hyperoctahedral Schur algebras.

Proposition 2.2.7. *For* $(\mathbf{i}, \mathbf{j})$, $(\mathbf{k}, \mathbf{l})$ *in* Ω, *we have*

$$\eta_{\mathbf{i},\mathbf{j}}\eta_{\mathbf{k},\mathbf{l}} = \sum_{(\mathbf{p},\mathbf{q})\in\Omega} Z(\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}, \mathbf{p}, \mathbf{q})\eta_{\mathbf{p},\mathbf{q}},$$

where $Z(\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}, \mathbf{p}, \mathbf{q}) := |\{\mathbf{s} \in I_B(2n, r) : (\mathbf{i}, \mathbf{j}) \sim (\mathbf{p}, \mathbf{s}) \text{ and } (\mathbf{k}, \mathbf{l}) \sim (\mathbf{s}, \mathbf{q})\}|$.

Proof. This is a consequence of the theory of permutation modules: see for example [**9**, Lemma II.12.8]. $\square$

2.3 Connection with general linear groups.

We note here an interesting connection between the algebra $H(2n, r)$ and a certain matrix group.

Definition 2.3.1. *We denote by* J *the element of* GL_{2n} *(over a given field) for which* J_{ab} *is* 1 *if* $a + b = 2n + 1$ *and* 0 *otherwise.*

We denote by $C(J)$ *the centraliser of* J *in* GL_{2n}. *This consists of all nonsingular matrices invariant under a rotation of the entries by a half turn.*

We index the rows and columns of the matrices by the ordered set

$$\{1, 2, \ldots, n, \bar{n}, \overline{n-1}, \ldots, \bar{1}\}.$$

Definition 2.3.2. *The group* $C(J)$ *acts on* $V_{2n}^{\otimes r}$ *via*

$$g.e_{\mathbf{j}} = \sum_{\mathbf{i}\in I_B(2n,r)} c_{\mathbf{i},\mathbf{j}}(g).e_{\mathbf{i}},$$

where

$$c_{\mathbf{i},\mathbf{j}}(g) := c_{i_1 j_1}(g) \cdots c_{i_r j_r}(g)$$

and $c_{ij}(g) := g_{ij}$.

Proposition 2.3.3. *The action of $C(J)$ on $V^{\otimes r}$ centralises the action of W given in §2.1.*

Proof. This works because $c_{\mathbf{i}.w,\mathbf{j}.w}(g) = c_{\mathbf{i},\mathbf{j}}(g)$ for all $w \in W$. (The only nontrivial aspect of this is the case where $w = s_r$.) $\square$

Note. This may be regarded as an analogue of the fact that in type A, the corresponding action of GL_n centralises the action of the symmetric group on tensor space.

3. The Hecke algebra and tensor space

3.1 Action of $\mathcal{H}(B_r)$ on tensor space.

The next aim is to generalise the results of §2 to the Hecke algebra $\mathcal{H}(B_r)$ corresponding to the Weyl group $W(B_r)$. We change the base field of vector spaces appearing from $\mathbb{Q}$ to $\mathbb{Q}(q, Q)$, where q and Q are independent indeterminates. (Contrast this to the situation in type A, which is "simply-laced", so only one indeterminate is required.) The integral forms of the objects appearing are now defined over the ring of Laurent polynomials $\mathcal{A} := \mathbb{Z}[q, q^{-1}, Q, Q^{-1}]$.

Definition 3.1.1. *The Hecke algebra $\mathcal{H}(B_r)$ is defined as the free $\mathcal{A}$-module with basis $\{T_w : w \in W(B_r)\}$ and the following multiplicative relations:*

$$T_{s_i} T_w = T_{s_i w} \quad \text{if } \ell(s_i w) > \ell(w),$$

$$T_{s_i} T_w = q T_{s_i w} + (q-1) T_w \quad \text{if } i < r, \ \ell(s_i w) < \ell(w),$$

$$T_{s_r} T_w = Q T_{s_r w} + (Q-1) T_w \quad \text{if } \ell(s_r w) < \ell(w).$$

Here, ℓ denotes the length function on the Coxeter group W.

Note that an equivalent definition (like the one in [**3**, §3]) would be to give the quadratic relations satisfied by the T_{s_i} and the braid relations of the Weyl group, similarly to the definition of the group W in §1. (This works because any two reduced words in W can be transformed into each other by means of a finite number of applications of the braid relations.) Also note that if the parameters q and Q are replaced by 1, one recovers the relations of the group algebra of W.

Lemma 3.1.2. *There is a right action of $\mathcal{H} = \mathcal{H}(B_r)$ on $V^{\otimes r}$ in which the generators T_{s_a} act on the basis $\{e_{\mathbf{i}} : \mathbf{i} \in I_B(2n, r)\}$ (where $\mathbf{i} = (i_1, \dots, i_r)$) as follows:*

$$e_{\mathbf{i}}.T_{s_a} = \begin{cases} qe_{\mathbf{i}.s_a} & \text{if } a < r,\ i_a \leq i_{a+1}, \\ e_{\mathbf{i}.s_a} + (q-1)e_{\mathbf{i}} & \text{if } a < r,\ i_a > i_{a+1}, \\ Qe_{\mathbf{i}.s_a} & \text{if } a = r,\ i_r \in \{1, \dots, n\}, \\ e_{\mathbf{i}.s_a} + (Q-1)e_{\mathbf{i}} & \text{if } a = r,\ i_r \in \{\bar{n}, \dots, \bar{1}\}. \end{cases}$$

Note. Note how we could have replaced $\leq$ by $<$, and $>$ by $\geq$ above, and the definition would have been the same.

Proof. Using the results in [**1**, Lemma 3.1.4], we know that the braid relations involving only the generators $T_{s_1}, \dots, T_{s_{r-1}}$ will hold, because they hold in type A. (Dipper and Donkin consider a left action, but this doesn't matter since the relations are left-right symmetric.)

The quadratic relation for T_{s_r} is easily checked and is similar to that for the other T_{s_i}.

The commutation relations corresponding to the commutations between s_r and s_i $(i < r)$ are clear.

The only nontrivial step is checking the relation $stst = tsts$, where we write s for T_{s_r} and t for $T_{s_{r-1}}$ for brevity. It is not hard to see that it is sufficient to check this in the case $r = 2$.

A short calculation shows that if i and j are both unbarred and $i \leq j$, then

$$(e_i \otimes e_j).stst = (e_i \otimes e_j).tsts = q^2 Q^2 (e_{\bar{i}} \otimes e_{\bar{j}}).$$

We must verify a similar statement in eight cases in total (one of which is the above calculation), according as i, j, neither, or both are barred, and according as $|i| \leq |j|$ or $|i| > |j|$. This can be deduced from the calculation in the above equation, together with the fact that there are eight equivalent formulations of the braid relation $stst = tsts$. These are the following:

$$stst = tsts,$$

$$s^{-1}tst = tsts^{-1},$$

$$s^{-1}t^{-1}st = tst^{-1}s^{-1},$$

$$s^{-1}t^{-1}s^{-1}t = ts^{-1}t^{-1}s^{-1},$$

$$s^{-1}t^{-1}s^{-1}t^{-1} = t^{-1}s^{-1}t^{-1}s^{-1},$$

$$st^{-1}s^{-1}t^{-1} = t^{-1}s^{-1}t^{-1}s,$$

$$sts^{-1}t^{-1} = t^{-1}s^{-1}ts,$$

$$stst^{-1} = t^{-1}sts.$$

Note that the use of inverses is valid, since we have verified all the quadratic relations and all the T_{s_i} are invertible. In each case, the application of the correct form of the above relation to $e_i \otimes e_j$ will yield the desired result very easily.

We illustrate this with an example. Consider $e_{\bar{i}} \otimes e_j$, where $i > j$, $\bar{i}$ is barred, and j is not. Then

$$e_{\bar{i}} \otimes e_j = q^{-1}Q^{-1}(e_j \otimes e_i).st.$$

Since $j < i$ and j and i are unbarred, we use the fact proved above that

$$(e_j \otimes e_i).stst = (e_j \otimes e_i).tsts$$

to show that

$$(e_{\bar{i}} \otimes e_j).sts^{-1}t^{-1} = (e_{\bar{i}} \otimes e_j).t^{-1}s^{-1}ts,$$

as required. The other cases are very similar to this. $\square$

Definition 3.1.3. *Let $w \in W(B_r)$. We write $q_w = Q^c q^{\ell(w)-c}$ where c is the number of occurrences of s_r in a reduced expression for w.*

Note. The number c is independent of the reduced expression chosen for w. This follows from the defining relations for W.

Lemma 3.1.4. *Let $\mathbf{a} = (a_1, \dots, a_r)$ represent an element of $w \in W(B_r)$. Then*

$$(e_1 \otimes \cdots \otimes e_r).T_w = q_w e_{a_1} \otimes \cdots \otimes e_{a_r}.$$

Proof. This follows by induction on the length of w, Lemma 1.2.2, Lemma 3.1.2 and Lemma 2.1.3. (The induction hypothesis is that if it holds for w, then it holds for ws_i where $\ell(ws_i) = \ell(w) + 1$.) $\square$

We may give a criterion to decide whether the above action of $\mathcal{H}$ is faithful, similar to the situation in type A.

Proposition 3.1.5. *The right action of $\mathcal{H}(B_r)$ on $V_{2n}^{\otimes r}$ is faithful if and only if $n \geq r$. The same is true for the action of the group algebra of $W(B_r)$ on $V_{2n}^{\otimes r}$.*

Proof. We prove the statement for $\mathcal{H}$. The statement for W may be obtained by replacing q and Q by 1 in the argument which follows.

Suppose $n \geq r$. We consider the dimension of the $\mathcal{H}$-module generated by the vector $e_1 \otimes \cdots \otimes e_r$. One finds from Lemma 3.1.4 that it is of dimension $|\mathbb{Z}_2 \wr \mathcal{S}_r|$, i.e. $2^r.r!$, from which it follows that the action is faithful.

Suppose that $n < r$. Consider the element $y \in \mathcal{H}$ given by

$$\sum_{w \in W} (-1)^{\ell(w)}.q_w^{-1} T_w.$$

By an induction on $\ell(w)$, we see that $T_w.y = (-1)^{\ell(w)} y$ for all $w \in W$; this follows by standard properties of Coxeter groups. Consider a typical basis element $e_{i_1} \otimes \cdots \otimes e_{i_r}$ of $V^{\otimes r}$. It follows from elementary properties of the action and Lemma 3.1.4 that this can be expressed in the form

$$e_{j_1} \otimes \cdots \otimes e_{j_r}.h$$

where $h \in \mathcal{H}$ (in fact, h may be taken to be a unit multiple of one of the basis elements of $\mathcal{H}$) and the indices j_m all come from the set $\{1, \ldots, n\}$ and satisfy $j_1 \leq \cdots \leq j_r$. We now observe from the theory of Coxeter groups that the element $y_i \in \mathcal{H}$ given by $T_1 - q_{s_i}^{-1} T_{s_i}$ divides y on the left. Since the element $e_{j_1} \otimes \cdots \otimes e_{j_r}$ contains a repeated index, one of the elements y_i for $1 \leq i < r$ acts as zero on it, and hence so do y and $h.y$. We deduce that y acts trivially on $V^{\otimes r}$, despite being nonzero. This completes the proof. $\square$

3.2 The centralising algebra of $\mathcal{H}(B_r)$.

We now generalise the results of §2.2 to the Hecke algebra situation. Not surprisingly, the results are more complicated.

Definition 3.2.1. *We denote by $H := H_{q,Q}(2n,r)$ the centralising algebra of $\mathcal{H}_{q,Q}(B_r)$ on $V_{2n}^{\otimes r}$. We denote its dual by $H^* := H^*_{q,Q}(2n,r)$.*

Proposition 3.2.2. *The coordinate functions $c_{\mathbf{i},\mathbf{j}}$ in $H^*_{q,Q}(2n,r)$ satisfy the following relations:*

$$c_{\mathbf{i},\mathbf{j}} = \begin{cases} qc_{\mathbf{i}.s_a,\mathbf{j}.s_a} & (a < r,\ i_a > i_{a+1},\ j_a \leq j_{a+1}) \\ c_{\mathbf{i}.s_a,\mathbf{j}.s_a} + (q-1)c_{\mathbf{i}.s_a,\mathbf{j}} & (a < r,\ i_a > i_{a+1},\ j_a > j_{a+1}) \\ c_{\mathbf{i}.s_a,\mathbf{j}.s_a} & (a < r,\ i_a = i_{a+1}) \\ Qc_{\mathbf{i}.s_r,\mathbf{j}.s_r} & (i_r \text{ barred, } j_r \text{ unbarred}) \\ c_{\mathbf{i}.s_r,\mathbf{j}.s_r} + (Q-1)c_{\mathbf{i}.s_r,\mathbf{j}} & (i_r \text{ barred, } j_r \text{ barred}) \end{cases}$$

Proof. The first three relations come from [**1**, Definition 1.1.1], and are necessary because $\mathcal{H}(B_r)$ contains the subalgebra $\mathcal{H}(A_{r-1})$ generated by $T_{s_1}, \ldots, T_{s_{r-1}}$, with the same action on tensor space.

We also need conditions corresponding to the fact that elements of $H_{q,Q}(2n,r)$ commute with the action of T_{s_r}. Since the action of T_{s_r} on an element $e_{\mathbf{j}}$ essentially involves only the entry j_r, we will abuse notation and only write out the last place in the tensor for clarity. Thus, T_{s_r} is thought of as acting on V itself.

Let η lie in the coordinate ring of $\mathrm{End}(V)$ and let j be unbarred, and write

$$\eta.e_j = \sum_{i \text{ unbarred}} (a_i e_i + b_i e_{\bar{i}}).$$

We know that $e_j.T_{s_r} = Qe_{\bar{j}}$, so if the actions of η and T_{s_r} on V commute, we find that

$$\eta.Qe_{\bar{j}} = (\eta.e_j).T_{s_r} = \sum_{i \text{ unbarred}} \big(b_i e_i + \big(Qa_i + (Q-1)b_i\big)e_{\bar{i}}\big).$$

Comparing coefficients proves that

$$c_{\bar{i}j} = Qc_{i\bar{j}}$$

and

$$c_{\bar{i}\bar{j}} = c_{ij} + (Q-1)c_{i\bar{j}},$$

where i and j are both unbarred. Knowing these facts makes the proof of the remaining relations easy. □

Definition 3.2.3. *Let $I_B^2 = I_B^2(2n, r)$ be that subset of $I_B(2n, r) \times I_B(2n, r)$ consisting of elements $(\mathbf{i}, \mathbf{j})$ satisfying the following two conditions:*

(i) *All the elements of $\mathbf{i}$ are unbarred, and $i_1 \leq \cdots \leq i_r$.*

(ii) *If $i_m = i_{m+1}$ then $j_m \leq j_{m+1}$.*

Remark. Notice that I_B^2 is in canonical bijection with monomials of degree r in $2n^2$ indeterminates, and therefore has cardinality

$$\binom{2n^2 + r - 1}{r}.$$

Also note that I_B^2 is a transversal for the W-orbits on $I_B(2n, r) \times I_B(2n, r)$.

Theorem 3.2.4. *The set $\{c_{\mathbf{i},\mathbf{j}} : (\mathbf{i}, \mathbf{j}) \in I_B^2\}$ is a basis for $H_{q,Q}^*(2n, r)$.*

Proof. First we show that the given set is a spanning set. Consider an arbitrary element $c_{\mathbf{a},\mathbf{b}}$. Using the first and second relations in Proposition 3.2.2, we can express this as a sum of elements $c_{\mathbf{k},\mathbf{l}}$ where $\mathbf{k} = (k_1, \ldots, k_r)$, $\mathbf{l} = (l_1, \ldots, l_r)$ and $k_1 \leq \cdots \leq k_r$, but where the k_m are not necessarily all unbarred. (In fact, the k_m are a rearrangement of the a_m.) If there are barred entries k_m, then k_r is one of them, and we apply the fourth or fifth relation according as l_r is unbarred or barred. This expresses $c_{\mathbf{a},\mathbf{b}}$ as a sum of elements $c_{\mathbf{a}',\mathbf{b}'}$ where all the $\mathbf{a}'$ have one fewer barred entry than $\mathbf{a}$. By iterating this process, we reduce to the case where $\mathbf{a}$ has no barred entries. After further applications of the first and second relations, we express $c_{\mathbf{a},\mathbf{b}}$ as a sum of elements satisfying condition (i). We then apply the third relation until condition (ii) is satisfied as well. This proves that the given set is a spanning set.

We now prove independence. Suppose

$$\sum_{(\mathbf{i},\mathbf{j}) \in I_B^2} \alpha_{\mathbf{i},\mathbf{j}} c_{\mathbf{i},\mathbf{j}} = 0 \tag{1}$$

for some scalars α. By clearing denominators and negative powers, we may assume all the α lie in $\mathbb{Z}[q, Q]$, and that the largest monomial involved is of minimal possible degree. Suppose that after substituting $Q = 1$, equation (1) becomes identically zero. This means that $(Q - 1)$ divides the equation over the field $\mathbb{Q}(q)$, and thus, by Gauss' Lemma, over $\mathbb{Z}[q]$, because the latter is a unique factorisation domain with field of fractions $\mathbb{Q}(q)$. This contradicts the minimality assumption, so equation (1) is nonzero after substituting $Q = 1$.

Now suppose we can find another equation satisfying the hypotheses of equation (1), but this time in the $\mathbb{Z}[q, q^{-1}]$-module $H^*_{q,1}(2n, r)$. If this becomes zero after setting $q = 1$, we may use a similar (but easier) argument to the above to show that $(q - 1)$ divides the equation. So if the equation is of minimal degree, it is nonzero after setting $q = Q = 1$.

This contradicts Corollary 2.2.4, since I_B^2 is a transversal of W orbits and the natural $\mathbb{Z}[q, q^{-1}, Q, Q^{-1}]$-form of $\mathcal{H}(B_r)$ becomes the natural $\mathbb{Z}$-form of $W(B_r)$ when q and Q are replaced by 1. This means that no equation of the form (1) exists, and that the given elements are independent. $\square$

One of the consequences of Theorem 3.2.4 is that it gives us a basis for H.

Definition 3.2.5. *The dual basis* $\{\eta_{\mathbf{i},\mathbf{j}} : (\mathbf{i}, \mathbf{j}) \in I_B^2\}$ *is a basis for* $H_{q,Q}(2n, r)$.

Proposition 3.1.5 tells us whether the right action of $\mathcal{H}$ on tensor space is faithful. We now show that the left action of $H = H_{q,Q}(2n, r)$ on tensor space is always faithful.

Remark. We end this section with a remark on the possibility of setting one or both of the parameters to 0. Because the relations on the $c_{\mathbf{i},\mathbf{j}}$ do not involve negative powers of q or Q, the basis of Theorem 3.2.4 still makes sense and is still a basis even when q and/or Q is set to zero. Using the comultiplication

$$\Delta(c_{\mathbf{p},\mathbf{q}}) = \sum_{\mathbf{s} \in I_B(2n,r)} c_{\mathbf{p},\mathbf{s}} \otimes c_{\mathbf{s},\mathbf{q}}.$$

(which is a q-analogue of the map Δ appearing in §2.2, even though it has exactly the same form), we find that the basis in Proposition 2.2.6 also makes sense at zero values of the parameters, and the algebra structure remains.

4. Quantized Permutation Modules

In §4, we consider certain natural "permutation" modules for the Hecke algebra $\mathcal{H}(B_r)$. It is important to do this in order to relate the algebra $H_{q,Q}(2n, r)$ to other algebras which appear in the literature.

4.1 Tensor space as a q-Permutation Module.

We start by defining a certain subset of parabolic subgroups of $W(B_r)$.

Definition 4.1.1. *We denote by $\mathcal{P}$ the set of parabolic subgroups generated by a set*

$$S \subseteq \{s_1, \dots, s_{r-1}\},$$

i.e. parabolic subgroups of W which do not involve the last reflection, s_r.

The next brace of definitions gives us the analogue of the theory of weight spaces in type A.

Definition 4.1.2. *Let $e = e_{i_1} \otimes \cdots \otimes e_{i_r} \in V^{\otimes r}$. Then the weight, $\lambda(e)$ of e is defined to be the n-tuple*

$$(\lambda_1, \dots, \lambda_r),$$

where λ_i is the total number of occurrences of the indices i and $\bar{i}$ in e. We denote the set of weights by $\Lambda_B = \Lambda_B(2n, r)$.

The element $\mathbf{u}_\lambda$ of $I_B(2n, r)$ is defined to be the r-tuple $(i_1, \dots, i_r)$, where $i_1 \leq \cdots \leq i_r \leq n$ and $e_{i_1} \otimes \cdots \otimes e_{i_r}$ has weight λ. We say an element of $I_B(2n, r)$ conjugate to $\mathbf{u}_\lambda$ (under the action of $W(B_r)$) has content λ.

The space V_λ is the span of all the basis vectors e of weight λ.

The vector e_λ is the unique basis vector $e \in V_\lambda$ whose indices satisfy $i_1 \leq \cdots \leq i_r \leq n$.

The subgroup $\mathcal{P}_\lambda$ is the element of $\mathcal{P}$ for which $s_i \in \mathcal{P}_\lambda$ if and only if the i-th and $(i+1)$-th entries of e_λ are equal.

The elements x_λ and y_λ of $\mathcal{H}$ are defined by

$$x_\lambda = \sum_{w \in \mathcal{P}_\lambda} T_w$$

and

$$y_\lambda = \sum_{w \in \mathcal{P}_\lambda} (-1)^{\ell(w)} q_w^{-1} T_w.$$

Lemma 4.1.3. *The space V_λ is a right $\mathcal{H}$-module.*

Proof. This is immediate from the action of $\mathcal{H}$ on V_λ which arises from the statement of Lemma 3.1.2. $\square$

Proposition 4.1.4. *As $\mathcal{H}$-modules, V_λ is isomorphic to $x_\lambda\mathcal{H}$. The isomorphism may be chosen to send e_λ to x_λ.*

Proof. It follows from Lemma 3.1.4 that the dimension of V_λ is equal to the index of $\mathcal{P}_\lambda$ (which is the stabiliser of e_λ), and that V_λ is generated as an $\mathcal{H}$-module by e_λ. We now set up a map from $x_\lambda\mathcal{H}$ to V_λ defined by

$$x_\lambda h \mapsto e_\lambda.h.$$

This is well-defined because x_λ acts as a scalar on e_λ. It is clearly surjective, and it is clearly a homomorphism of right $\mathcal{H}$-modules. Injectivity follows by a comparison of dimensions: the dimension of $x_\lambda\mathcal{H}$ is also the index of the parabolic subgroup $\mathcal{P}_\lambda$. □

4.2 Endomorphism Rings of Quantized Permutation Modules.

Next we compare our algebra with certain algebras appearing in [**6**].

Lemma 4.2.1. *There is an automorphism of $\mathcal{H}$ taking x_λ to y_λ for all λ. Therefore there is an algebra isomorphism*

$$\mathrm{End}_{\mathcal{H}}\left(\bigoplus_{\mathcal{P}_\lambda\in\mathcal{P}} x_\lambda\mathcal{H}\right) \cong \mathrm{End}_{\mathcal{H}}\left(\bigoplus_{\mathcal{P}_\lambda\in\mathcal{P}} y_\lambda\mathcal{H}\right).$$

Proof. The required automorphism is a generalisation of that in [**2**, 2.1]. It sends T_{s_i} to $-q_{s_i}T_{s_i}^{-1}$; it is straightforward to check that this is an automorphism. We see from [**2**, 2.1] that x_λ and y_λ are exchanged by this automorphism.

A standard argument now shows that

$$\mathrm{Hom}_{\mathcal{H}}(x_\lambda\mathcal{H}, x_\mu\mathcal{H}) \cong \mathrm{Hom}_{\mathcal{H}}(y_\lambda\mathcal{H}, y_\mu\mathcal{H}),$$

from which the lemma follows. □

Proposition 4.2.2. *The algebra $H_{q,Q}(2n,r)$ is Morita equivalent to the algebra $S(\mathcal{H})$ of [**6**] in the case where $\mathcal{H} = \mathcal{H}(B_r)$.*

Proof. The algebra $S(\mathcal{H})$ is defined in [**6**, §7.2] as

$$\mathrm{End}_{\mathcal{H}}\left(\bigoplus_{\mathcal{P}_\lambda\in\mathcal{P}} y_\lambda\mathcal{H}\right).$$

It follows easily from the definitions that

$$H_{q,Q}(2n,r) \cong \mathrm{End}_{\mathcal{H}}\left(\bigoplus_{\lambda\in\Lambda_B} V_\lambda\right).$$

A standard argument shows that this is Morita equivalent to

$$H'_{q,Q}(2n,r) := \mathrm{End}_{\mathcal{H}}\left(\bigoplus_{\mathcal{P}_\lambda\in\mathcal{P}} V_\lambda\right)$$

The proof now follows from Proposition 4.1.4 and Lemma 4.2.1. $\square$

4.3 Du's Generalised Schur Algebras.

In §4.3, we assume that $Q = q$, partly for compatibility with the Kazhdan–Lusztig polynomials, $P_{y,w}(q)$, where $y, w \in W$.

The following definition comes from [**4**].

Definition 4.3.1 (Du). *The generalised Schur algebra of type B_r is given by*

$$\mathcal{S}_q(W(B_r)) = \mathrm{End}_{\mathcal{H}}\left(\bigoplus_{\mathcal{P}_\lambda\in\mathcal{P}'} x_\lambda\mathcal{H}\right),$$

where $\mathcal{P}'$ is the set of all parabolic subgroups of W.

This is a free $\mathbb{Z}[q,q^{-1}]$ module with basis $\phi^d_{I,J}$ as I and J range over $\mathcal{P}'$ and d is a distinguished double W_I–W_J coset respresentative.

The effect of $\phi^d_{I,J}$ on $\bigoplus_{J'\in\mathcal{P}'} x_{J'}\mathcal{H}$ is given by

$$\phi^d_{I,J} : x_{J'}T_w \mapsto \delta_{J,J'}T_{W_I dW_J}T_w,$$

where T_C in general stands for $\sum_{w\in C} T_w$.

Proposition 4.3.2. *The elements $\phi^d_{I,J}$ appearing in Definition 4.3.1 which satisfy $I, J \in \mathcal{P}$ span a subalgebra isomorphic to the algebra $H'_{q,q}(2n,r)$ appearing in Proposition 4.2.2.*

Proof. This follows immediately from Definition 4.3.1 and the proof of Proposition 4.2.2. $\square$

Remark. Du also defines a basis $\theta^d_{I,J}$, ranging over the same indexing set as the ϕ basis. This also has the property that the elements for which $I, J \in \mathcal{P}$ form a basis for the algebra H'. This basis is related to the ϕ basis via the Kazhdan–Lusztig polynomials.

It is convenient to extend the basis $\phi^d_{I,J}$ of $H'_{q,q}(2n,r)$ to one of $H_{q,q}(2n,r)$. This is achieved as follows.

Definition 4.3.3. *The algebra $\widetilde{S}_q(2n,r)$ is defined to be*

$$\mathrm{End}_{\mathcal{H}}\left(\bigoplus_{\lambda\in\Lambda_B(2n,r)} x_\lambda\mathcal{H}\right).$$

This is a free $\mathbb{Z}[q,q^{-1}]$ module with basis $\phi^d_{\lambda,\mu}$ for all $\lambda,\mu\in\Lambda_B(2n,r)$ and d a distinguished double $\mathcal{P}_\lambda$–$\mathcal{P}_\mu$ coset respresentative.

The effect of $\phi^d_{\lambda,\mu}$ on $\bigoplus_{\lambda\in\Lambda_B} x_\nu\mathcal{H}$ is given by

$$\phi^d_{\lambda,\mu} : x_\nu T_w \mapsto \delta_{\mu,\nu} T_{\mathcal{P}_\lambda d\mathcal{P}_\mu} T_w.$$

Lemma 4.3.4. *We have*

$$\widetilde{S}_q(2n,r)\cong H_{q,q}(2n,r).$$

Proof. This is clear from the proof of Proposition 4.2.2. □

4.4 Agreement of bases.

A natural question to ask is whether the basis $\eta_{\mathbf{i},\mathbf{j}}$ is compatible with the basis $\phi^d_{\lambda,\mu}$. This was proved in type A by Dipper and Donkin (see [**1**, Theorem 3.2.5]). It turns out to be true in type B as well, and the techniques in [**1**] are extendable to type B. We continue to assume that $Q=q$ in §4.4.

First, we need to find an identification between the two bases, but this turns out not to be difficult.

Lemma 4.4.1. *There is a bijection between W-orbits of $I_B(2n,r)\times I_B(2n,r)$ and basis elements $\phi^d_{\lambda,\mu}$ of $H_{q,Q}(2n,r)$ which maps $\phi^d_{\lambda,\mu}$ to the orbit containing the element $(\mathbf{u}_\lambda.d,\mathbf{u}_\mu)$.*

Proof. The proof follows from an elementary argument. □

We can now prove a result analogous to [**1**, Lemma 3.2.3].

Lemma 4.4.2. *Consider $c_{\mathbf{i},\mathbf{j}}$, where $(\mathbf{i},\mathbf{j})$ is an element of I_B^2, and $\mathbf{i}$ and $\mathbf{j}$ have contents λ and μ respectively. Let d be a distinguished right coset representative for $\mathcal{P}_\lambda$ in W such that $(\mathbf{i},\mathbf{j}) \sim (\mathbf{i}.d, \mathbf{u}_\mu)$. Then $c_{\mathbf{i}.d,\mathbf{u}_\mu} = q_d c_{\mathbf{i},\mathbf{j}}$, and*

$$c_{\mathbf{i}.d,\mathbf{u}_\mu}(\eta_{\mathbf{i},\mathbf{j}}) = \begin{cases} q_d & \text{if } (\mathbf{i}.d, \mathbf{u}_\mu) \sim (\mathbf{i},\mathbf{j}), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We apply a "straightening procedure" to the element $c_{\mathbf{i}.d,\mathbf{u}_\mu}$ to express it as a sum of basis elements. Each step of this involves replacing an element $c_{\mathbf{a},\mathbf{b}}$ by a combination of others by using one of the five rules of Proposition 3.2.2.

As in the discussion following [**1**, Corollary 3.2.2], the fact that

$$\mathbf{u}_\mu = (u_1, \dots, u_r),$$

where $u_1 \leq \cdots \leq u_r \leq n$ means that during the straightening procedure, we never have to apply the second and fifth relations, which are the ones that produce extra terms. (This is a consequence of Lemma 1.2.2.) A multiple of q appears for each adjacent pair of indices sorted into order on the $\mathbf{i}$-side, and a multiple of Q appears for each entry on the $\mathbf{i}$-side as it is unbarred. We know from Lemma 1.2.3 that each simple reflection in a fixed reduced expression of d will cause one of these events to happen, so by the time the straightening is complete, we end up with

$$c_{\mathbf{i}.d,\mathbf{u}_\mu} = q_d c_{\mathbf{i},\mathbf{j}}.$$

The second assertion now follows immediately. $\square$

Definition 4.4.3. *Let $W_I, W_J \in \mathcal{P}$ and $w \in W$. We define the subset*

$$\mathcal{C}(W_I, w, W_J)$$

of W to be the set of distinguished right coset representatives for W_I in W which lie in the double coset $W_I w W_J$. Thus we have

$$W_I w W_J = \dot{\bigcup_{d \in \mathcal{C}(W_I, w, W_J)}} W_I d.$$

Theorem 4.4.4. *As an endomorphism, $\eta_{\mathbf{i},\mathbf{j}}$ is equal to $\phi^d_{\lambda,\mu}$, under the identifications given in Lemma 4.4.1.*

Proof. Let λ and μ be the contents of $\mathbf{i}$ and $\mathbf{j}$, respectively. We wish to show that

$$\eta_{\mathbf{i},\mathbf{j}}.x_\mu = T_{W_I d W_J} = \sum_{w \in \mathcal{C}(W_I, w, W_J)} x_\lambda.T_w.$$

Now we use the identification of Proposition 4.1.4, and the fact that $e_{\mathbf{u}_\mu}.T_w = q_w e_{\mathbf{u}_\mu.w}$; the latter follows easily by induction. This shows that the coefficient of $e_{\mathbf{a}}$ in $\eta_{\mathbf{i},\mathbf{j}} e_{\mathbf{u}_\mu}$ is q_d if $(\mathbf{a}, \mathbf{u}_\mu) \sim (\mathbf{i}, \mathbf{j})$ (where $\mathbf{a} = \mathbf{i}.d$ and d is a distinguished right coset representative for $\mathcal{P}_\lambda$) and 0 otherwise. Lemma 4.4.2 now completes the proof. □

4.5 The Double Centraliser Property.

This set-up makes it relatively easy to prove a double centraliser property analogous to Schur–Weyl duality in type A. We assume $n \geq r$ throughout §4.5.

Lemma 4.5.1. *Any endmorphism of $V^{\otimes r}$ which commutes with the action of $H_{q,Q}(2n, r)$ is determined by its effect on the vector*

$$e_1 \otimes \cdots \otimes e_r.$$

Proof. Let ω be the weight

$$(\underbrace{1, 1, \ldots, 1}_{r}, 0, 0, \ldots, 0).$$

The set

$$\{\phi^d_{\lambda,\omega}.e_\omega : \lambda \in \Lambda_B, d \in \mathcal{D}_\lambda\}$$

(where $\mathcal{D}_\lambda$ is the set of distinguished right coset representatives of $\mathcal{P}_\lambda$) is a basis for V_λ. (This is essentially just another formulation of Lemma 3.1.4, using Proposition 4.1.4.) The statement now follows from the fact that the actions of H and $\mathcal{H}$ on $V^{\otimes r}$ commute. □

Lemma 4.5.2. *Each element $\eta_{\mathbf{i},\mathbf{i}}$ in H is idempotent and has the property that*

$$\eta_{\mathbf{i},\mathbf{i}}.e_{\mathbf{j}} = \begin{cases} e_{\mathbf{j}} & \text{if } \mathbf{i} \sim \mathbf{j}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The conditions

$$c_{\mathbf{a},\mathbf{b}}(\eta) = \begin{cases} 1 & \text{if } \mathbf{a} = \mathbf{b} \sim \mathbf{i}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbf{a}$ and $\mathbf{b}$ range over all elements of $I_B(2n, r)$, satisfy the relations on the $c_{\mathbf{a},\mathbf{b}}$ given in Proposition 3.2.2. Furthermore, the only element $c_{\mathbf{a},\mathbf{b}}$ with $(\mathbf{a}, \mathbf{b}) \in I_B^2$ which does not annihilate η is $c_{\mathbf{i},\mathbf{i}}$, and this takes value 1 on η. Thus $\eta = \eta_{\mathbf{i},\mathbf{i}}$. The action of η on $e_{\mathbf{j}}$ is now clearly as claimed, from which it follows that η is idempotent. □

Lemma 4.5.3. *Let ψ be an endomorphism of $V^{\otimes r}$ which commutes with the action of H. Then, writing $\psi(e_{\mathbf{i}}) = \sum_{\mathbf{k}} \alpha_{\mathbf{k}} e_{\mathbf{k}}$, we have $\alpha_{\mathbf{k}} \neq 0 \Rightarrow \mathbf{k} \sim \mathbf{i}$.*

Proof. Suppose $\alpha_{\mathbf{k}} \neq 0$ for some $\mathbf{k} \not\sim \mathbf{i}$. Let $\mathbf{j}$ be such that $\mathbf{j} \sim \mathbf{k}$ and $\mathbf{j} = (j_1, \ldots, j_r)$ where $j_1 \leq \cdots \leq j_r$ and all the j_m are unbarred. Then by Lemma 4.5.2, $\eta_{\mathbf{j},\mathbf{j}}.e_{\mathbf{k}} = e_{\mathbf{k}}$, from which it follows that $\eta_{\mathbf{j},\mathbf{j}}(\psi(e_{\mathbf{i}})) \neq 0$. However, $\eta_{\mathbf{j},\mathbf{j}}(e_{\mathbf{i}}) = 0$ because $\mathbf{i} \not\sim \mathbf{j} \sim \mathbf{k}$. It is therefore not possible that the action of $\eta_{\mathbf{j},\mathbf{j}}$ commutes with that of ψ, which is a contradiction. □

Proposition 4.5.4. *The algebras $H_{q,Q}(2n, r)$ and $\mathcal{H}_{q,Q}(B_r)$ are centralising algebras of each other on tensor space.*

Proof. Firstly, note that by Proposition 3.1.5, the action of each algebra is faithful, since the action of H is faithful by construction.

We know that H is the centralising algebra of $\mathcal{H}$, by construction of H. Lemma 4.5.3 shows that if ψ commutes with the action of $\mathcal{H}$, then it maps each weight space V_λ to itself. Lemma 4.5.1 now shows that the centralising algebra of H is of dimension at most $\dim \mathcal{H}$. Conversely, since $\mathcal{H}$ acts faithfully on the weight space containing the tensor $e_1 \otimes \cdots \otimes e_r$, this bound is achieved, which completes the proof. □

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