# ON 321-AVOIDING PERMUTATIONS IN AFFINE WEYL GROUPS

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Abstract. We introduce the notion of 321-avoiding permutations in the affine Weyl group $W$ of type $A_{n-1}$ by considering the group as a George group (in the sense of Eriksson and Eriksson). This enables us to generalize a result of Billey, Jockusch and Stanley to show that the 321-avoiding permutations in $W$ coincide with the set of fully commutative elements; in other words, any two reduced expressions for a 321-avoiding element of $W$ (considered as a Coxeter group) may be obtained from each other by repeated applications of short braid relations.

Using Shi's characterization of the Kazhdan–Lusztig cells in the group $W$, we use our main result to show that the fully commutative elements of $W$ form a union of Kazhdan–Lusztig cells. This phenomenon has been studied by the author and J. Losonczy for finite Coxeter groups, and is interesting partly because it allows certain structure constants for the Kazhdan–Lusztig basis of the associated Hecke algebra to be computed combinatorially.

We also show how some of our results can be generalized to a larger group of permutations, the extended affine Weyl group associated to $GL_n(\mathbb{C})$.

**To appear in the Journal of Algebraic Combinatorics**

## Introduction

Let $W$ be a Coxeter group with generating set $S = \{s_i\}_{i \in I}$. The fully commutative elements of $W$ were defined by Stembridge, and may be characterised [**18**, Proposition 1.1] as those elements all of whose reduced expressions avoid strings of length $m$ of the form $s_i s_j s_i \cdots$, where $2 < m = m_{i,j}$ is the order of $s_i s_j$. Billey, Jockusch and Stanley [**1**, Theorem 2.1] show that in the case where $W$ is the symmetric group (under the natural identifications), $w \in W$ is fully commutative if and

2000 *Mathematics Subject Classification.* 05E15, 20C32.
This paper was written at the Newton Institute, Cambridge, and supported by the special semester on Symmetric Functions and Macdonald Polynomials, 2001.

only if it is 321-avoiding. The latter condition means that there are no $i < j < k$ such that $w(i) > w(j) > w(k)$. It is clear from properties of root systems of type $A$ that the 321-avoiding elements $w \in W$ are precisely those for which there are no positive roots $\alpha, \beta$ such that both (i) $\alpha + \beta$ is a root and (ii) $w(\alpha)$ and $w(\beta)$ are both negative roots.

In this paper we consider the affine analogue of this problem. To define the analogue of a 321-avoiding permutation, we recall Lusztig's realization of the affine Weyl group of type $\widetilde{A}_{n-1}$ as a group of permutations of the integers. The interplay between the permutation representation and the root system of type $\widetilde{A}$ can be seen in the work of Papi [**15**], and the relationship between full commutativity and root systems in simply laced types has been established by Fan and Stembridge [**6**]. It is therefore not hard to show that the 321-avoiding elements and the fully commutative elements coincide in type $\widetilde{A}$, but we choose to give a different derivation of this result (avoiding root systems) because the equivalence of conditions (iii) and (iv) of Theorem 2.7 is not made explicit in [**15**].

Our real interest in this problem stems from a desire to understand the Kazhdan–Lusztig basis combinatorially. The two-sided Kazhdan–Lusztig cells for Coxeter groups of type $\widetilde{A}_{n-1}$ were classified by Shi [**16**] according to a combinatorial criterion on elements $w \in W$ generalizing the notion of 321-avoidance. Although it may be possible to classify the Kazhdan–Lusztig cells of a given Coxeter group using a purely algebraic or combinatorial approach, it is not reasonable to expect that the Kazhdan–Lusztig basis $\{C'_w : w \in W\}$ itself and its structure constants may be understood using similar means. However, as we discuss in §5, if the fully commutative elements form a union of Kazhdan–Lusztig cells closed under $\geq_{LR}$, results of the author and J. Losonczy in [**10**] reduce the problem of computing the coefficient of $C'_z$ in $C'_x C'_y$ to a tractable combinatorial problem in the case that $z$ is fully commutative.

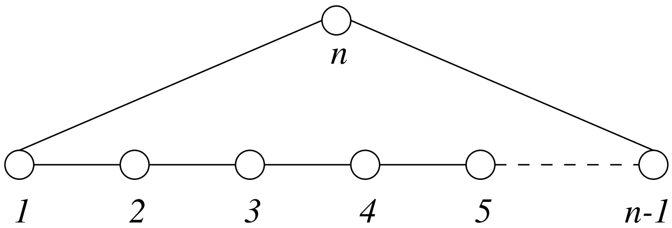
We also show how these results may be generalized to a certain extension of the affine Weyl group of type $\widetilde{A}_{n-1}$.

## 1. Affine Weyl groups of type $A$ as permutations

In §1, we show how the affine Weyl group $W$ of type $A$ and its extension $\widehat{W}$ may be viewed as groups of permutations of $\mathbb{Z}$. In the case of the Coxeter group $W$, Eriksson and Eriksson [**3**, §5] call such a group of permutations a "George group" (after G. Lusztig). The group $\widehat{W}$ is called (following [**19**, §2.1]) the extended affine Weyl group associated to $GL_n(\mathbb{C})$, and it can be similarly considered as a group of permutations. We follow the treatment of these groups in [**7**, §1] but work with left actions rather than right actions.

The affine Weyl group of type $\widetilde{A}_{n-1}$ arises from the Dynkin diagram in Figure 1.

FIGURE 1. Dynkin diagram of type $\widetilde{A}_{n-1}$




The number of vertices in Figure 1 is $n$. The top vertex is regarded as an extra relative to the remainder of the graph, which is a Coxeter graph of type $A_{n-1}$. We will assume $n \geq 3$ throughout this paper.

We associate an affine Weyl group, $W = W(\widetilde{A}_{n-1})$, to this Dynkin diagram in the usual way (as in [**12**, §2.1]). This associates to node $i$ of the graph a generating involution $s_i$ of $W$, where $s_i s_j = s_j s_i$ if $i$ and $j$ are not adjacent in the graph, and

$$s_i s_j s_i = s_j s_i s_j$$

if $i$ and $j$ are adjacent in the graph. For $t \in \mathbb{Z}$, it is convenient to denote by $\bar{t}$ the congruence class of $t$ modulo $n$, taking representatives in the set $\{1, 2, \dots, n\}$. For the purposes of this paper, it is helpful to think of the group $W$ as follows.

**Proposition 1.1 (Lusztig).** *There exists a group isomorphism from $W$ to the set of permutations of $\mathbb{Z}$ which satisfy the following conditions:*

$$w(i+n) = w(i) + n \text{ for } i \in \mathbb{Z}, \tag{1}$$

$$\sum_{t=1}^{n} w(t) = \sum_{t=1}^{n} t \tag{2}$$

*such that $s_i$ is mapped to the permutation*

$$t \mapsto \begin{cases} t & \text{if } \bar{t} \neq \bar{i}, \overline{i+1}, \\ t-1 & \text{if } \bar{t} = \overline{i+1}, \\ t+1 & \text{if } \bar{t} = \bar{i}, \end{cases}$$

*for $t \in \mathbb{Z}$.*

*Proof.* This result is stated in [**14**, §3.6]; a proof may be found in [**3**, Theorem 20]. □

We also define an extension, $\widehat{W}$, of $W$ as a group of permutations of $\mathbb{Z}$, as in [**19**]. This group will be discussed in more depth in §4.

**Definition 1.2.** Let $\rho$ be the permutation of $\mathbb{Z}$ taking $t$ to $t+1$ for all $t$. Then the group $\widehat{W}$ is defined to be the group of permutations of $\mathbb{Z}$ generated by the group $W$ and $\rho$.

**Proposition 1.3.** *There exists a group isomorphism from $\widehat{W}$ to the set of permutations of $\mathbb{Z}$ that satisfies the condition $w(i+n) = w(i)$ for all $i \in \mathbb{Z}$. Any element of $\widehat{W}$ is uniquely expressible in the form $\rho^z w$ for $z \in \mathbb{Z}$ and $w \in W$. Conversely, any element of this form is an element of $\widehat{W}$.*

*Proof.* Note that we automatically have

$$\sum_{t=1}^{n} w(t) \equiv \sum_{t=1}^{n} t \mod n$$

because $w$ is a permutation. The result now follows from [**7**, Proposition 1.1.3, Corollary 1.1.4]. □

The group $\widehat{W}$ can be expressed in a more familiar way using a semidirect product construction.

**Proposition 1.4.** *Let $\mathcal{S}_n$ be the subgroup of $\widehat{W}$ generated by*

$$\{s_1, s_2, \dots, s_{n-1}\}.$$

*Let $Z$ be the subgroup of $\widehat{W}$ consisting of all permutations $z$ satisfying*

$$z(t) \equiv t \mod n$$

*for all $t$. Then $\mathbb{Z}^n \cong Z \trianglelefteq \widehat{W}$ and $W$ is isomorphic to the semidirect product of $\mathcal{S}_n$ and $Z$.*

*Proof.* See [**7**, Proposition 1.1.5]. □

In the language of [**3**, 5.1.2], the George group $W$ is defined as the group of "locally finite" permutations which commute with all elements of the "rigid group" of translations generated by $\rho^n$. Proposition 1.4 is saying that the group $\widehat{W}$ may be obtained by dropping the hypothesis of local finiteness.

It is convenient to extend the usual notion of the length of an element of a Coxeter group to the group $\widehat{W}$ in the following way.

**Definition 1.5.** Let $w \in W$ and let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a word in the set $S$ of generating involutions such that $r$ is minimal. Then we define the length of $w$, $\ell(w)$, to be equal to $r$.

The length, $\ell(w')$, of a typical element $w' = \rho^z w$ of $\widehat{W}$ (where $z \in \mathbb{Z}$ and $w \in W$) is defined to be $\ell(w)$.

Definition 1.5 is very natural from the point of view of George groups: in the language of [**3**, §6.1] it states that the length of an element $w \in \widehat{W}$ is equal to the "class inversion number" of $w$.

## 2. Pattern avoidance in Coxeter groups

In §2, we restrict our attention to the Coxeter group $W$ of type $\widetilde{A}_{n-1}$. We first recall the notions of full commutativity and 321-avoidance.

**Definition 2.1.** Let $w \in W$.

We say $w$ is *fully commutative* if no reduced expression for $w$ contains a subword of the form $s_i s_j s_i$ for Coxeter generators $s_i$ and $s_j$. The set of fully commutative elements of $W$ is denoted by $W_c$.

We say $w$ is 321-avoiding if, when $W$ is considered as a George group as in Proposition 1.1, there are no integers $a < b < c$ such that $w(a) > w(b) > w(c)$.

*Remark 2.2.* Notice that if $w \in W$ is such that $a < b < c$ and $w(a) > w(b) > w(c)$, it must be the case that $w(a)$, $w(b)$ and $w(c)$ are pairwise non-congruent modulo $n$.

The following result will be used to address the issue of checking whether an infinite permutation is 321-avoiding.

**Proposition 2.3.** *Let $w \in W$ and $a, b, c \in \mathbb{Z}$ with $a < b < c$. If $w(a) > w(b) > w(c)$, then there exist $a', c' \in \mathbb{Z}$ with $0 < b - a' < n$, $0 < c' - b < n$, $a \equiv a' \mod n$, $c \equiv c' \mod n$, $a' < b < c'$ and $w(a') > w(b) > w(c')$.*

*Proof.* First, note that $a$, $b$ and $c$ are pairwise noncongruent modulo $n$ by Remark 2.2.

Write $c = c' + kn$ with $k \in \mathbb{Z}^{\geq 0}$ and $0 < c' - b < n$. By condition (1) of Proposition 1.1, we may assume without loss of generality that $k = 0$ by replacing the triple $(a, b, c)$ by $(a, b, c')$. This satisfies the claimed properties because $w(b) > w(c) = w(c - kn) + kn \geq w(c')$.

The argument to find $a'$ is similar, starting with the condition $a = a' - kn$ with $k \in \mathbb{Z}^{\geq 0}$ and $0 < b - a' < n$. $\square$

*Remark 2.4.* Suppose $w \in W$ is given as a permutation. Proposition 2.3 shows that we can determine whether there exist $a < b < c$ with $w(a) > w(b) > w(c)$ by testing $\binom{n}{3}$ triples. This is because if such a triple exists, we may assume that $1 \leq b \leq n$ by condition (1) of Proposition 1.1, and we may apply Proposition 2.3 to show that there exists a triple that in addition satisfies $b - a < n$ and $c - b < n$. (Recall that $a, b, c$ will be pairwise noncongruent modulo $n$ by Remark 2.2.)

The next result will be useful in the proof of Theorem 2.7.

**Lemma 2.5.**

(a) *Let $w \in W$. We have*

$$w(i) < w(i+1) \Leftrightarrow \ell(ws_i) > \ell(w)$$

*and*

$$w^{-1}(i) < w^{-1}(i+1) \Leftrightarrow \ell(s_i w) > \ell(w).$$

(b) *If $c, d \in \mathbb{Z}$ with $c < d$, and $s \in S$ is a generator of $W$, then $s(c) \geq s(d)$ implies that $c = d - 1$, $s(c) = d$ and $s(d) = c$.*

*Proof.* Part (a) is a restatement of [**7**, Corollary 1.3.3].

It is clear from Proposition 1.1 that $s(c) \geq s(d)$ can only occur if $d - c \leq 2$, because $|s(z) - z| \leq 1$ for all $z \in \mathbb{Z}$. If $d - c = 1$ and $s(c) \geq s(d)$, then the fact that $s$ induces a permutation of $\mathbb{Z}$ implies that $s(c) = d$ and $s(d) = c$ as claimed. If $d - c = 2$ and $s(c) \geq s(d)$, we must have $s(c) = c + 1$ and $s(d) = d - 1$, which also cannot happen since $s$ induces a permutation and $c + 1 = d - 1$. Part (b) follows. $\square$

The following result is reminiscent of [**2**, Lemma 1].

**Lemma 2.6.** *Let $w \in W$ be fully commutative, and let $s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression for $w$. Suppose $s_{i_j}$ and $s_{i_k}$ are consecutive occurrences of the generator $s$ (meaning that $s_{i_j} = s = s_{i_k}$ for some $j < k$ and $s_{i_l} \neq s$ for all $j < l < k$). Let $t$ and $t'$ be the two distinct Coxeter generators that do not commute with $s$. Then there is an occurrence $s_{i_p} = t$ and an occurrence $s_{i_q} = t'$ with $j < p, q < k$.*

*Note.* This result is a consequence of [**5**, Lemma 4.3.5 (ii)] and its proof, but here we present a more direct argument based on property R3 of [**4**, §2].

*Proof.* Since $w \in W_c$, we can see that between the two named occurrences of $s$ there must be at least two occurrences of generators that do not commute with $s$. If we choose $s$ such that $k - j$ is minimal, there must be at least one occurrence of each of the generators $t$ and $t'$ in the given interval, as required. □

**Theorem 2.7.** *Let $w \in W = W(\widetilde{A}_{n-1})$. The following are equivalent:*

(i) *$w$ is fully commutative (considering $W$ as a Coxeter group);*

(ii) *if $a, b \in \mathbb{Z}$ with $a < b$ and we have $w(a) > w(b)$, then $w(a) > a$ and $w(b) < b$;*

(iii) *$w$ is 321-avoiding (considering $W$ as a George group);*

(iv) *there are no positive roots $\alpha, \beta, \alpha + \beta$ in the root system of type $\widetilde{A}_{n-1}$ such that $w(\alpha) < 0$ and $w(\beta) < 0$.*

*Note.* Recall that the action of a generator $s_i \in W$ on a simple root $\alpha_j$ is given by

$$s_i.\alpha_j = \begin{cases} -\alpha_j & \text{if } i = j, \\ \alpha_j + \alpha_i & \text{if } \bar{i} = \overline{j-1} \text{ or } \bar{i} = \overline{j+1}, \\ \alpha_j & \text{otherwise.} \end{cases}$$

The reader is referred to [**6**] or [**15**] for further details of the root system.

*Proof.* The equivalence of (i) and (iv) follows from [**6**, Theorem 2.4].

(i) $\Rightarrow$ (ii).

Suppose that (ii) fails to hold. Suppose $a < b$ and $w(a) > w(b)$ and $w(a) \leq a$. (The other possibility, that $w(b) \geq b$, is proved using an argument similar to the following.)

Let $s_{i_r} s_{i_{r-1}} \cdots s_{i_1}$ be a reduced expression for $w$. For $1 \leq j \leq r$, define

$$w[j] := s_{i_j} s_{i_{j-1}} \cdots s_{i_1},$$

with $w[0] := 1$ for notational convenience. Since $a < b$ and $w(a) > w(b)$, there must exist $1 \leq j \leq r$ such that $w[j](a) \geq w[j](b)$ and $w[j-1](a) < w[j-1](b)$.

Applying Lemma 2.5 (b) with $c = w[j-1](a)$, $d = w[j-1](b)$ and $s = s_{i_j}$, we find that $w[j](b) = w[j-1](a) = w[j-1](b) - 1 = w[j](a) - 1$. There are two cases to check; they cover the possibilities because $w(a) \leq a$.

Case 1: $w(a) < w[j](a)$. Now $w(a) = w[r](a) < w[j](a) > w[j-1](a)$. It follows that there exist $k$ and $l$ with $j \leq k < l \leq r$ such that

$$w[k-1](a)+1 = w[k](a) = w[k+1](a) = \cdots = w[l-2](a) = w[l-1](a) = w[l](a)+1.$$

It follows that $s_{i_k} = s_{i_l}$ and that no generator $s_{i_p}$ for $k < p < l$ is equal to $s_{i_k}$ or to $s_{1+i_k}$ (where addition is taken modulo $n$). Applying Lemma 2.6 to $s_{i_k}$ and $s_{i_l}$ now shows that $w$ is not fully commutative, because $s_{1+i_k}$ does not commute with $s_{i_k}$.

Case 2: $a > w[j-1](a)$. Here $w[0](a) = a > w[j-1](a) < w[j](a)$. It follows that there exist $k$ and $l$ with $1 \leq k < l \leq j$ and

$$w[k-1](a)-1 = w[k](a) = w[k+1](a) = \cdots = w[l-2](a) = w[l-1](a) = w[l](a)-1.$$

The consideration of this case is now similar to case 1, mutatis mutandis.

(ii) $\Rightarrow$ (iii).

Suppose (iii) fails, so that there exist $a < b < c$ with $w(a) > w(b) > w(c)$. If $w(b) \leq b$ then (ii) fails when applied to $b < c$, and if $w(b) \geq b$ then (ii) fails when applied to $a < b$.

(iii) $\Rightarrow$ (i).

Suppose (i) fails. Let $s_{i_r} s_{i_{r-1}} \cdots s_{i_1}$ be a reduced expression for $w$ for which $s_{i_{t+1}} = s_{i_{t-1}}$ for some $1 < t < r$. Define $w[j]$ as in the proof of (i) $\Rightarrow$ (ii) above. Let us assume that $i_t = i_{t+1} + 1$; the other case ($i_t = i_{t+1} - 1$) follows by a symmetric argument.

Let $y = w[t-2]$ and $m = i_{t-1}$ (so that $m + 1 = i_t$ by the previous paragraph). We first claim that $y^{-1}(m) < y^{-1}(m+1) < y^{-1}(m+2)$. Since $s_{i_{t-1}} y > y$, Lemma 2.5 (a) shows that $y^{-1}(m) < y^{-1}(m+1)$. Since $\ell(s_{i_{t+1}} s_{i_t} s_{i_{t-1}} y) = \ell(y) + 3$ and $s_{i_{t+1}} s_{i_t} s_{i_{t-1}} = s_{i_t} s_{i_{t-1}} s_{i_t}$, it follows that $s_{i_t} y > y$ and Lemma 2.5 (a) shows that $y^{-1}(m+1) < y^{-1}(m+2)$. (This uses the fact that $i_t = m+1$.)

Define $a = y^{-1}(m)$, $b = y^{-1}(m+1)$ and $c = y^{-1}(m+2)$, and let $z = w[t+1]$. It follows from the definition of the realization of the generators of $W$ as permutations (given in Proposition 1.1) and the above paragraph that $z(a) = m+2$, $z(b) = m+1$ and $z(c) = m$. We are done once we show that $w(a) > w(b) > w(c)$. Suppose

$w(a) \leq w(b)$. Then there exists $k$ such that $t+1 < k \leq r$ where we have $w[k-1](a) > w[k-1](b)$ and $w[k](a) < w[k](b)$. This implies that $w[k-1](a) = w[k-1](b) + 1$ by Lemma 2.5 (b). Since $\ell(s_{i_k} w[k-1]) > \ell(w[k-1])$, applying Lemma 2.5 (a) with $i = w[k-1](b)$ yields a contradiction. We deduce that $w(a) > w(b)$ and the proof that $w(b) > w(c)$ is similar. $\square$

It is possible to prove the equivalence of conditions (iii) and (iv) of Theorem 2.7 by using the formalism of George groups from [**3**]. If $w \in W$ then the condition that $w\alpha < 0$ for a positive root $\alpha$ is equivalent to the condition that $\ell(wt) < \ell(w)$, where $t$ is the (not necessarily simple) reflection corresponding to the root $\alpha$. The simple reflections are defined in Proposition 1.1 and reflections in general will be conjugates of these, so they are easily identified. A more direct approach using roots as opposed to reflections is implicit in [**15**, Theorem 1].

It would be interesting to have an interpretation of condition (ii) of Theorem 2.7 in terms of roots, since this would enable the condition to be generalized to other types. Condition (ii) is also interesting in that it captures the notion of 321-avoiding by considering the images of only two elements.

## 3. Kazhdan–Lusztig cells

We wish to apply Theorem 2.7 in order to understand better the Kazhdan–Lusztig cells in type $\widetilde{A}_{n-1}$. For any Coxeter group $W$, Kazhdan and Lusztig [**13**] defined partitions of $W$ into left cells, right cells and two-sided cells, where the two-sided cells are unions of left (or right) cells. These cells are naturally partially ordered by an explicitly defined order, $\leq_{LR}$. We do not present the full definitions here; an elementary introduction to the theory may be found in [**12**, §7].

The properties of these cells are subtle and it seems impossible to understand them fully without using heavy machinery such as intersection cohomology. However, the problem of classifying the two-sided cells is combinatorially tractable in special cases, and Shi [**16**] succeeded in describing combinatorially the two-sided cells for the Coxeter group of type $\widetilde{A}_{n-1}$. One of the main results of [**16**] describes a natural bijection between two-sided cells and partitions of $n$. This bijection, which is a generalization of the Robinson–Schensted correspondence, depends on concepts of which 321-avoidance is a special case.

Our aim in §3 is to explain why the fully commutative elements in type $\widetilde{A}$ form a union of Kazhdan–Lusztig cells closed under the order $\geq_{LR}$. Fan and Stembridge [**6**, Theorem 3.1] have shown that for Coxeter groups of types $A$, $D$, $E$ and $\widetilde{A}$, $W_c$ is a union of Spaltenstein–Springer–Steinberg cells. It is not always true (even in the aforementioned cases) that $W_c$ is a union of Kazhdan–Lusztig cells, but the situation for finite Coxeter groups is well understood: the author and J. Losonczy proved in [**10**, Corollary 3.1.3] that an irreducible finite Coxeter group has this property if and only if it does not contain a parabolic subgroup of type $D_4$.

We start by recalling Shi's map $\sigma$ from $W$ to the partitions of $n$ as described in [**16**]. (We have changed the action of $W$ on $\mathbb{Z}$ from being a right action to being a left action, but this makes no difference and $\sigma(w) = \sigma(w^{-1})$.)

**Definition 3.1.** Fix $w \in W = W(\widetilde{A}_{n-1})$. Define, for $k \geq 1$,

$$d_k = d_k(w) := \max \left\{ |X| : X = \bigcup_{i=1}^{k} X_i \subset \mathbb{Z} \right\}$$

where (a) the union is disjoint, (b) any two integers $u, v \in X$ are required not to be congruent modulo $n$, and (c) whenever the subset $X_i$ of $\mathbb{Z}$ contains integers $u$ and $v$ with $u < v$ then we require $w(u) > w(v)$. (Here, $|X|$ denotes the cardinality of $X$.)

The partition $\sigma(w)$ of $n$ is defined by

$$(d_1, d_2 - d_1, d_3 - d_2, \dots, d_t - d_{t-1}),$$

where $t$ is maximal subject to the condition that $d_{t-1} < d_t$ (meaning that $d_t = n$).

It may not be obvious from the definition that $\sigma(w)$ is a partition (i.e., $d_1 \geq d_2 - d_1 \geq \cdots \geq d_t - d_{t-1}$), but this follows by a result of Greene and Kleitman [**11**].

Recall that if $\lambda$ and $\mu$ are two partitions of $n$, we say that $\lambda \trianglerighteq \mu$ ($\lambda$ dominates $\mu$) if for all $k$ we have

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i.$$

This defines a partial order on the set of partitions of $n$. As mentioned before, there is also a natural partial order $\leq_{LR}$ on the two-sided Kazhdan–Lusztig cells of a Coxeter group. The following theorem explains how these two orders are related.

**Theorem 3.2 (Shi).** *Let $y, w \in W = W(\widetilde{A}_{n-1})$. Then $y \leq_{LR} w$ in the sense of Kazhdan–Lusztig if and only if $\sigma(y) \trianglerighteq \sigma(w)$. In particular, $y$ and $w$ are in the same two-sided cell (i.e., $y \leq_{LR} w \leq_{LR} y$) if and only if $\sigma(y) = \sigma(w)$.*

*Proof.* See [**17**, §2.9]. □

As we now explain, it is not hard to see from these definitions that the 321-avoiding elements of $W$ are a union of Kazhdan–Lusztig cells.

**Lemma 3.3.** *Fix $w \in W$.*

(i) *If $w$ is 321-avoiding then all the sets $X_i$ in Definition 3.1 satisfy $|X_i| \leq 2$.*
(ii) *If $w$ is not 321-avoiding then the number $d_1$ in Definition 3.1 satisfies $d_1 \geq 3$.*

*Proof.* If there is a set $X_i$ with $|X_i| \geq 3$ then by definition of $X_i$, there exist $u_1, u_2, u_3 \in X_i$ with $u_1 < u_2 < u_3$ but $w(u_1) > w(u_2) > w(u_3)$. This proves (i). The proof of (ii) is similar: if $a < b < c$ with $w(a) > w(b) > w(c)$ then we may choose $X_1$ to contain these integers $a, b, c$ (and possibly others), thus showing that $d_1 \geq 3$. □

**Theorem 3.4.** *Let $W_c$ denote the set of fully commutative elements of $W(\widetilde{A}_{n-1})$. Then $W_c$ is a union of two-sided Kazhdan–Lusztig cells closed under $\geq_{LR}$.*

*Note.* By the statement $W_c$ is closed under $\geq_{LR}$ we mean that if $w \in W_c$ and $y \geq_{LR} w$ then $y \in W_c$.

*Proof.* It is clearly enough to show that if $w \in W \backslash W_c$ and $w \geq_{LR} y$ then $y \in W \backslash W_c$. By theorems 2.7 and 3.2, it is enough to show that if $w$ is not 321-avoiding and $\sigma(y) \trianglerighteq \sigma(w)$ then $y$ is not 321-avoiding.

Suppose $w$ and $y$ are as above. Let $\lambda = \sigma(w)$ and $\mu = \sigma(y)$. By Lemma 3.3 (ii), $\lambda_1 \geq 3$. Since $\mu \trianglerighteq \lambda$, it follows from the definition of the dominance order that $\mu_1 \geq 3$. By Lemma 3.3 (i), this means that $y$ is not 321-avoiding, as required. □

We will discuss some of the applications of Theorem 3.4 in §5.

## 4. The extended affine Weyl group associated to $GL_n(\mathbb{C})$

We now return to the group $\widehat{W}$ of Proposition 1.3. The generalization of Theorem 2.7 to this group is straightforward as we now show. Recall that every element $w' \in \widehat{W}$ may be written uniquely as $\rho^z w$ for $z \in \mathbb{Z}$ and $w \in W$, where $\rho$ is as in Definition 1.2.

**Definition 4.1.** Let $w' = \rho^z w$ be a typical element of $\widehat{W}$. We say $w'$ is *fully commutative* if $w$ is fully commutative, and we say $w'$ is 321-*avoiding* if $w$ is 321-avoiding.

The definition of 321-avoiding above is a natural one as if $a < b$ then $w(a) < w(b)$ if and only if $w'(a) < w'(b)$.

The group $\widehat{W}$ may be made to act on the root system of $W$ in a natural way which is clear from the following fact.

**Lemma 4.2.** *Let $s_i$ be a Coxeter generator of $W$. Then $\rho s_i \rho^{-1} = s_{i+1}$, where addition is taken modulo $n$.*

*Proof.* This follows by considering the induced permutations on $\mathbb{Z}$. □

We define the action of $\widehat{W}$ on the root system by stipulating that $\rho(\alpha_i) = \alpha_{i+1}$ (with addition taken modulo $n$); this is reasonable by Lemma 4.2. Notice that $\rho^n$ acts as the identity; the extended affine Weyl group associated to $SL_n(\mathbb{C})$ is the quotient of $\widehat{W}$ by the relation $\rho^n = 1$.

**Proposition 4.3.** *Let $w \in \widehat{W}$. The following are equivalent:*

(i) *$w$ is fully commutative;*
(ii) *$w$ is 321-avoiding;*
(iii) *there are no positive roots $\alpha, \beta, \alpha + \beta$ in the root system of type $\widetilde{A}_{n-1}$ such that $w(\alpha) < 0$ and $w(\beta) < 0$.*

*Proof.* The equivalence of (i) and (ii) is immediate from the definitions and Theorem 2.7. It follows from the definition of the action of $\rho$ on simple roots that powers of $\rho$ permute the positive roots linearly. This means that if $w = \rho^z w'$ then $w'(\alpha) < 0$ and $w'(\beta) < 0$ if and only if $w(\alpha) < 0$ and $w(\beta) < 0$. The equivalence of (iii) with the other conditions now follows by Theorem 2.7. □

There seems to be no direct analogue of condition (ii) of Theorem 2.7 for the group $\widehat{W}$.

Our results here suggest the following question about George groups.

**Question 4.4.** *Is it possible to define in a uniform way an extended George group for affine types $A$, $B$, $C$ and $D$ that specializes to the group $\widehat{W}$ in type $A$?*

We do not necessarily expect that the extended George group for types $B$, $C$ and $D$ should be closely related to the group $\widehat{W}$.

It is also natural to wonder whether Theorem 3.4 may be extended to other affine types. The theorem will fail for Coxeter systems containing a parabolic subgroup of type $D_4$ by the results of [**10**], which rules out type $\widetilde{D}$ and type $\widetilde{B}_l$ (at least for $l \geq 4$). The most interesting open case is therefore that of type $\widetilde{C}$.

## 5. Applications to Kazhdan–Lusztig theory

Theorem 3.4 shows that for the group $W = W(\widetilde{A}_{n-1})$, the set $W_c$ is a union of two-sided Kazhdan–Lusztig cells. This result may be refined as follows.

**Proposition 5.1.** *Let $W_c$ denote the set of fully commutative elements of $W(\widetilde{A}_{n-1})$. Then the number of distinct two-sided cells contained in $W_c$ is $(n+1)/2$ if $n$ is odd, and $(n+2)/2$ if $n$ is even. A set of representatives in $W$ for the two-sided cells is given by the set*

$$\left\{ s_2 s_4 \cdots s_{2k} : 0 \leq k \leq \frac{n}{2} \right\},$$

*using the usual numbering of the generators shown in Figure 1.*

*Proof.* By theorems 2.7, 3.2, 3.4 and Lemma 3.3, the number of fully commutative two-sided cells in $W_c$ is equal to the number of partitions of $n$ with all parts less than or equal to 2, and the first claim follows. It follows from Definition 3.1 that if $w = s_2 s_4 \cdots s_{2k}$, the partition $\sigma(w)$ has $k$ parts equal to 2 and the other parts equal to 1. The result follows. □

We end by discussing briefly the application of Theorem 3.4 to the computation of certain structure constants for the Kazhdan–Lusztig basis.

The Kazhdan–Lusztig basis, which first appeared in [**13**], is a free $\mathbb{Z}[v, v^{-1}]$-basis for the Hecke algebra $\mathcal{H}(W)$ associated to the group $W$. The basis, $\{C'_w : w \in W\}$, is naturally indexed by $W$. The structure constants $g_{x,y,z}$, namely the Laurent polynomials occurring in the expression

$$C'_x C'_y = \sum_{z \in W} g_{x,y,z} C'_z,$$

have many subtle properties, such as the fact that $g_{x,y,z} \in \mathbb{N}[v, v^{-1}]$. (It seems impossible to establish this fact combinatorially.)

Theorem 3.4 implies that the $\mathbb{Z}[v, v^{-1}]$-span of the set

$$J(W) = \{C'_w : w \notin W_c\}$$

is an ideal of $\mathcal{H}(W)$. The quotient algebra $\mathcal{H}(W)/J(W)$ is equipped with a natural basis $\{c_w : w \in W_c\}$, where $c_w := C'_w + J(W)$. The basis $\{c_w : w \in W_c\}$ is the canonical basis (in the sense of [**9**]) for the quotient of the Hecke algebra $\mathcal{H}(W)/J(W)$ by [**10**, Theorem 2.2.3]. Although we do not present full details, it is possible to realize this canonical basis combinatorially; the description is similar to that given in [**8**, Definition 6.4.3]. This means that, if $z \in W_c$, we can compute $g_{x,y,z}$ by simple combinatorial means. (Note that if $z \in W_c$ and $g_{x,y,z} \neq 0$ then we must have $x, y \in W_c$.) In particular, it is easily checked that $g_{x,y,z} \in \mathbb{N}[v, v^{-1}]$ in this case.

## Acknowledgements

The author thanks J.R. Stembridge for some helpful discussions and J. Losonczy for comments on the manuscript.

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