CONSTRUCTING CELL DATA FOR DIAGRAM ALGEBRAS

R.M. Green and P.P. Martin

Department of Mathematics
University of Colorado
Campus Box 395
Boulder, CO 80309-0395
USA
E-mail: rmg@euclid.colorado.edu

Centre for Mathematical Science
City University
Northampton Square
London EC1V 0HB
England
E-mail: P.P.Martin@city.ac.uk

Abstract. We show how the treatment of cellularity in families of algebras arising from diagram calculi, such as Jones' Temperley–Lieb wreaths, variants on Brauer's centralizer algebras, and the contour algebras of Cox et al (of which many algebras are special cases), may be unified using the theory of tabular algebras. This improves an earlier result of the first author (whose hypotheses covered only the Brauer algebra from among these families).

Introduction

Cellular algebras were introduced by Graham and Lehrer [**10**], and are a class of finite dimensional associative algebras defined in terms of a "cell datum" and three axioms. The axioms allow one to define a set of modules for the algebra known as "cell modules", and one of the main strengths of the theory is that it is relatively

1991 *Mathematics Subject Classification.* 16G30.

straightforward to construct and to classify the irreducible modules for a cellular algebra in terms of quotients of the cell modules.

Tabular algebras were introduced by the first author in [**13**] as a class of associative $\mathbb{Z}[v, v^{-1}]$-algebras equipped with distinguished bases (tabular bases) and satisfying certain axioms. In the most general setting, tabular algebras are defined via a somewhat complicated "table datum" extending the cell datum construct. However, there is a large natural subclass, the so-called "tabular algebras with trace", which may be defined (up to isomorphism, in a sense made precise in [**15**]) simply by giving the distinguished basis. In [**16**], the first author introduced "cell modules" and "standard modules" for tabular algebras; each of these classes of modules is analogous in some sense to the cell modules of a cellular algebra.

The motivation for the theory of tabular algebras is twofold. On one hand, the theory can be viewed as a framework for studying the properties of "canonical bases" for algebras. The latter objects have been studied abstractly using constructions such as Du's IC bases [**8**] and Stanley's P-kernels [**31**], and the archetypal examples are the celebrated Kazhdan–Lusztig bases introduced in [**20**]. This is the point of view taken in [**16**], where it is shown that the Kazhdan–Lusztig bases of certain extended affine Hecke algebras of type A are tabular bases, and that the standard modules for the tabular algebra agree with the geometrically defined standard modules appearing in the work of Lusztig [**24**] and others.

The other main motivation for the theory of tabular algebras is as templates for cellular algebras. Core to this is a theorem [**13**, Theorem 2.1.1] giving conditions under which one can describe a cellular structure for an algebra in terms of the tabular structure. As we will explain in this paper, there are many cases in the literature where a cellular basis for an algebra has been constructed in terms of another basis that turns out to be tabular; in these situations, the tabular algebras may thus be regarded as more basic objects than the corresponding cellular algebras. This is a helpful point of view because tabular bases have some advantages over cellular bases: they are defined integrally (over $\mathbb{Z}[v, v^{-1}]$), and they are easy to construct in many cases because they occur naturally in the contexts of Kazhdan–Lusztig type bases and algebras given by diagram calculi (many of which arise, for example, in computational statistical mechanics [**25**, **29**, **33**]).

The main purpose of this paper is to generalize the core theorem [**13**, Theorem 2.1.1] to cover a wider class of examples, with particular emphasis on examples that arise from algebras given by a calculus of diagrams. We thus obtain shorter proofs of cellularity in several of the known examples of cellular algebras (see for example corollaries 4.2.4, 5.2.3, and 5.3.6): the constructions are similar, but our approach has the advantage of relative generality. One can also use our main result, Theorem 2.2.1, to construct new examples of cellular algebras. One way to do this is using Theorem 4.1.3, which shows how to construct a kind of wreath product of certain cellular algebras, which in turn yields new examples of cellular algebras. Another method we use involves the notion of a subdatum (introduced in Definition 2.1.4), which is a convenient way to describe certain subalgebras of tabular algebras as tabular algebras in their own right (Proposition 2.1.5). It may be anticipated that these techniques will provide further short proofs of cellularity in future applications.

Diagram Algebras and Cellular Algebras

The formal definition of *diagram algebra* is beyond the scope of this paper (see [**28**] for the core paradigm), but there are some simple components which it will be useful to bring to mind, in order most simply to complete discussion of the historical context of our work.

Let X be a poset. A *formal diagram category* (on X) is a category whose objects are X, and morphisms $d \in \mathrm{Hom}(x, y)$ are called diagrams, with properties (some of) which are described below. If $d \in \mathrm{Hom}(x, y)$ may be expressed as $d_1 d_2$ with $d_1 \in \mathrm{Hom}(x, x')$ and $d_2 \in \mathrm{Hom}(x', y)$ we say d *factors* through x'. A *propagating* *index* of d is a lowest element of X such that d factors through it. Such is not unique in general: we let $\#d$ denote the set of propagating indices of d. A diagram category has the *filtration property*

$$\#d_1 d_2 \leq \#d_1,\ \#d_2$$

i.e. $x \in \#d_1 d_2$ implies $x \leq y$ for $y \in \#d_i$.

If $d \in \mathrm{Hom}(x,x)$ has x as a propagating index it is said to be *flush*. In particular, 1_x is flush. The subset of flush diagrams is denoted $\mathrm{Hom}_t(x,x)$. In a diagram category the composite of flush diagrams is flush, so $\Gamma(x) = \mathrm{Hom}_t(x,x)$ is a kind of submonoid of $\mathrm{Hom}(x,x)$.

Let us focus for a moment on the observation that $\mathrm{Hom}(x,y)$ has an action of $\mathrm{Hom}(x,x)$ on the left and $\mathrm{Hom}(y,y)$ on the right. (So far this set has only a left semigroup action of $\mathrm{Hom}(x,x)$ rather than a module structure, but it will be convenient to adopt (bi)module terminology.) This $\mathrm{Hom}(x,y)$ can be partitioned into components with given propagating index, and the parts with propagating indices less than a given index (z say) form a sub-bimodule, by the filtration property. Denote the quotient $\mathrm{Hom}^z(x,y)$. Suppose that $x \leq y$. Then (in a diagram category) $\mathrm{Hom}^x(x,y)$ is isomorphic to a sum of copies of $\mathrm{Hom}_t(x,x)$ with respect to its left action. This paper concerns (from the diagram algebra perspective) the inheritance of properties of $\mathrm{Hom}_t(x,x)$–modules by the x-section of the above filtration (again, see [**28**] for concrete examples).

Diagram algebras and cellular algebras have a history of intertwined development, and this inheritance aspect is no exception. The first several notable examples of cellular bases appeared *before* the introduction of cellular algebras, in diagram algebra contexts (see [**17**, **26**, **29**]). In the physical contexts in which diagram algebras occur (e.g. as transfer matrix algebras) the cellular axioms which we will describe in the next section correspond heuristically, but closely, to ideas of information propagation and of spatial or time-reversal symmetry. On the other hand, König and Xi [**21**] have introduced their 'inflationary' construction for cellular algebras, which puts these ideas in a nice abstract setting. A diagram algebra is an algebra with a 'natural' basis of diagrams. As noted above, each diagram may be cut into two parts: a top and a bottom, say. The set of possible top parts is taken into the set of bottom parts by an operation i turning them upside down. This set is partitioned into subsets ("by) layers" $a \sim b$ if a, $i(b)$ can be the parts of a cut diagram. However the combination of a, $i(b)$ is not in general unique, this combination being controlled by an intermediate algebra. Thus layers of the algebra take the form $V \otimes G \otimes V$ where G is the intermediate.

König and Xi noted that subject to some technical conditions consistent with the above, a rather general free R–module V (irrespective of diagrams) may be used with a cellular algebra G to produce another cellular algebra with layer $V \otimes G \otimes V$. The argument hinges on an equivalent definition of cellular algebra that does not use bases, and the construction does not provide any 'inflated' basis.

In practical matters of representation theory of concrete algebras, however, explicit cellular (indeed any) bases are invaluable. Here, using the first author's relatively robust "tabular bases" and algebras G which are hypergroups we are able to develop a version of inflation with bases. As we will see, these bases can be chosen to *be* the natural bases in diagram algebra examples. Thus our theorem offers a formalism well tuned to analysis of such concrete algebras, a useful counterpoint to König and Xi's elegantly abstract construction.

1. Cellular algebras and hypergroups

1.1 Cellular algebras.

Cellular algebras were originally defined by Graham and Lehrer [**10**].

Definition 1.1.1. Let R be a commutative ring with identity. A *cellular algebra* over R is an associative unital algebra, A, together with a cell datum $(\Lambda, M, C, *)$ where:

(C1) Λ is a finite poset. For each $\lambda \in \Lambda$, $M(\lambda)$ is a finite set such that

$$C : \coprod_{\lambda \in \Lambda} (M(\lambda) \times M(\lambda)) \to A$$

is injective with image an R-basis of A.

(C2) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, we write $C(S,T) = C^\lambda_{S,T} \in A$. Then $*$ is an R-linear involutory anti-automorphism of A such that $(C^\lambda_{S,T})^* = C^\lambda_{T,S}$.

(C3) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ then for all $a \in A$ we have

$$a.C^\lambda_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^\lambda_{S',T} \mod A(< \lambda),$$

where $r_a(S', S) \in R$ is independent of T and $A(< \lambda)$ is the R-submodule of A generated by the set

$$\{C^\mu_{S'',T''} : \mu < \lambda, S'' \in M(\mu), T'' \in M(\mu)\}.$$

Remark 1.1.2. We have assumed Λ to be finite to avoid complications (see [**11**, §1.2]).

We now recall from the literature some of the main examples of cellular of algebras that are particularly relevant for our purposes in this paper.

Example 1.1.3. Let $\mathcal{S}_n$ be the symmetric group on n letters. Then the group algebra $\mathbb{Z}\mathcal{S}_n$ is cellular over $\mathbb{Z}$. In this case, the poset Λ is the set of partitions of n, ordered by dominance (meaning that if $\lambda \trianglerighteq \mu$ then $\lambda \leq \mu$). The set $M(\lambda)$ is the set of standard tableaux of shape λ, namely the ways of writing the numbers $1, \dots, n$ once each into a Young diagram of shape λ such that the entries increase along rows and down columns. The element $C^\lambda_{S,T}$ is the Kazhdan–Lusztig basis element C_w such that $w \in \mathcal{S}_n$ corresponds via the Robinson–Schensted correspondence to the ordered pair of standard tableaux (S, T). The map $*$ sends C_w to $C_{w^{-1}}$.

The Hecke algebra $\mathcal{H}(\mathcal{S}_n)$ was shown to be cellular by Graham and Lehrer in [**10**, Example 1.2], and the underlying idea was already implicit in [**20**]. The example of the symmetric group above is obtained simply by specializing q to 1, as was observed by Graham and Lehrer in their treatment of the Brauer algebra [**10**, §4] For details on the relationship between the Robinson–Schensted correspondence and Kazhdan–Lusztig theory, the reader is referred to Ariki's paper [**3**].

Example 1.1.4. A simple example of a cellular algebra that is important for our purposes is Graham and Lehrer's so-called "banal example" [**10**, Example 1.3]. Let R be a commutative ring with identity, let $\lambda_1, \lambda_2, \dots, \lambda_k$ be (not necessarily distinct) elements of R, and let $P(x) \in R[x]$ be the polynomial $\prod_{i=1}^{k}(x - \lambda_i)$. Then the rank k algebra $A = R[x]/\langle P(x) \rangle$ is cellular over R. A cell datum is as follows: Λ is the poset $\{1, 2, \dots, k\}$, ordered in the natural way, $M(\lambda)$ is a one-element set for each λ and $C^j_{S,S}$ is the image of the polynomial $\prod_{i=j+1}^{k}(x - \lambda_i)$. The map $*$ is the identity map, which is an anti-automorphism because A is commutative.

Let A and A' be cellular algebras over R with cell data $(\Lambda_1, M_1, C_1, *_1)$ and $(\Lambda_2, M_2, C_2, *_2)$ respectively. We will show in the next two examples how the direct sum and direct product of two cellular algebras are again cellular in a natural way. (We omit the proofs of these results because they are both well-known and easy: see the remarks at the end of [**21**, §6].) Later (Theorem 4.1.3) we will look at a less trivial way to form new cellular algebras using a kind of wreath product.

Example 1.1.5. Let Λ_3 be the disjoint union of Λ_1 and Λ_2. We partially order order Λ_3 by stipulating that $\lambda \leq \lambda'$ if, for some $i \in \{1, 2\}$, we have $\lambda, \lambda' \in \Lambda_i$ and $\lambda \leq_i \lambda'$, where $\leq_i$ is the partial order on Λ_i. For $\lambda \in \Lambda_3$, we define $M_3(\lambda)$ to be $M_1(\lambda)$ if $\lambda \in \Lambda_1$ and $M_2(\lambda)$ if $\lambda \in \Lambda_2$. We define C_3 (respectively, $*_3$) in an analogous way as a natural extension of C_1 and C_2 (respectively, $*_1$ and $*_2$). The R-algebra $A \oplus A'$ is then cellular with cell datum $(\Lambda_3, M_3, C_3, *_3)$.

Example 1.1.6. Let Λ_4 be the Cartesian product $\Lambda_1 \times \Lambda_2$, partially ordered by stipulating that $(\lambda_1, \lambda_2) \leq_4 (\lambda'_1, \lambda'_2)$ if and only if $\lambda_1 \leq_1 \lambda'_1$ and $\lambda_2 \leq_2 \lambda'_2$. For $\lambda = (\lambda_1, \lambda_2) \in \Lambda_4$, we define $M_4(\lambda)$ to be $M_1(\lambda_1) \times M_2(\lambda_2)$. For $(S_1, S_2), (T_1, T_2) \in M(\lambda)$, we define $C_4((S_1, S_2), (T_1, T_2))$ to be $C_1(S_1, T_1) \otimes_R C_2(S_2, T_2)$. The map $*_4$ is the R-linear map sending $C_4((S_1, S_2), (T_1, T_2))$ to $C_4((T_1, T_2), (S_1, S_2))$. The R-algebra $A \otimes_R A'$ is then cellular with cell datum $(\Lambda_4, M_4, C_4, *_4)$.

1.2 Hypergroups.

A key ingredient in the definition of tabular algebras is the notion of a hypergroup. There are many variants of this idea in the literature, for example the table algebras of Arad and Blau [**1**], the generalized table algebras of Arad, Fisman and Muzychuk [**2**], the association schemes of algebraic combinatorics [**4**] and the discrete hypergroups as described by Sunder [**32**, Definition IV.1]. The hypergroups we define here are the "normalized table algebras" of [**13**, Definition 1.1.2], but we use the name "hypergroups" here for simplicity and to reflect the fact that most of the important examples of hypergroups considered in this paper are in fact groups.

Definition 1.2.1. A hypergroup is a pair (A, B), where A is an associative unital R-algebra for some $R \geq \mathbb{Z}$ and $B = \{b_i : i \in I\}$ is a distinguished basis for A such that $1 \in B$, satisfying the following three axioms:

(H1) The structure constants of A with respect to the basis B are nonnegative integers.

(H2) There is an algebra anti-automorphism $^-$ of A whose square is the identity and that has the property that $b_i \in B \Rightarrow \overline{b_i} \in B$. (We define $\bar{i}$ by the condition $\overline{b_i} = b_{\bar{i}}$.)

(H3) Let $\kappa(b_i, a)$ be the coefficient of b_i in $a \in A$. Then we have $\kappa(b_m, b_i b_j) = \kappa(b_i, b_m \overline{b_j})$ for all $i, j, m \in B$.

Remark 1.2.2. Note that, setting $b_m = 1$ in axiom (H3), we see that $b_{\bar{i}}$ can be characterized by the property that it is the unique basis element b_j for which 1 appears with nonzero coefficient in $b_i b_j$ (or in $b_j b_i$). This implies that the anti-automorphism $^-$ is completely determined by the structure constants.

Example 1.2.3. Perhaps the most obvious example of a hypergroup is the case where B is a group G and A is the group algebra RG, where R is a ring containing $\mathbb{Z}$. In this case, the anti-automorphism $^-$ is the R-linear extension of inversion in G.

The following result is an easy consequence of axiom (H3) (see also [**14**, Proposition 1.1.4]).

Proposition 1.2.4. *Let (A, B) be a hypergroup. The linear function t sending $a \in A$ to $\kappa(1, a)$ satisfies $t(xy) = t(yx)$ for all $a \in A$.* $\square$

The following result shows how a tensor product of two hypergroups is another hypergroup. (In the case of groups, this construction corresponds to the direct product.)

Proposition 1.2.5. *Let (A_1, B_1) and (A_2, B_2) be hypergroups over R. Then*

$$(A_1 \otimes_R A_2, B_1 \otimes_R B_2)$$

is a hypergroup over R, where the multiplication on $A_1 \otimes A_2$ is given by the Kronecker product and the anti-automorphism $^-$ of $A_1 \otimes_R A_2$ is defined to send $b_1 \otimes_R b_2$ to $\overline{b_1} \otimes_R \overline{b_2}$.

Proof. See, for example, [**14**, Proposition 1.1.5]. $\square$

1.3 Based rings.

We now recall Lusztig's notion of a based ring; see [**23**] or [**36**, §1.5].

Definition 1.3.1. A *based ring* is a pair (A, B), where A is a unital $\mathbb{Z}$-algebra with free $\mathbb{Z}$-basis B and nonnegative structure constants. A homomorphism $\phi : (A, B) \longrightarrow (A', B')$ of based rings is a homomorphism of abstract $\mathbb{Z}$-algebras $\phi : A \longrightarrow A'$ such that $\phi(b) \in B' \cup \{0\}$ for all $b \in B$. Isomorphisms, automorphisms, anti-automorphisms, etc. of based rings are defined analogously.

Remark 1.3.2. Clearly hypergroups over $\mathbb{Z}$ give examples of based rings, and the map $^-$ of axiom (H2) is an anti-automorphism of based rings.

Proposition 1.3.3. *Let $(A, \mathbf{B})$ and $(A', \mathbf{B}')$ be hypergroups over $\mathbb{Z}$ and let*

$$f : (A, \mathbf{B}) \longrightarrow (A', \mathbf{B}')$$

be a unital homomorphism of based rings. Then, for each $a \in A$, we have $f(\bar{a}) = \overline{f(a)}$, where the anti-automorphisms $\bar{\ }$ are those associated by Definition 1.2.1 to each hypergroup.

Proof. Because f is $\mathbb{Z}$-linear, we may immediately reduce our consideration to the case where $a \in \mathbf{B}$.

Let $b_i \in \mathbf{B}$, and consider the equation

$$b_i b_{\bar{i}} = 1 + \sum_{1 \neq b_k \in \mathbf{B}} c_k b_k,$$

which holds by Remark 1.2.2. Applying f to the equation and using the hypothesis that f sends $1 \in \mathbf{B}$ to $1 \in \mathbf{B}'$, we obtain

$$f(b_i) f(b_{\bar{i}}) = 1 + \sum_{1 \neq b_k \in \mathbf{B}} c_k f(b_k).$$

Although it may be the case that $f(b_k) = 1$ for some $b_k \neq 1$, axiom (H1) shows that the coefficient of 1 on the right hand side is nonzero. Remark 1.2.2 now shows that $f(b_{\bar{i}}) = f(\overline{b_i}) = \overline{f(b_i)}$, where the first equality is by definition of $b_{\bar{i}}$. $\square$

Proposition 1.3.3 ensures that the following definition makes sense.

Definition 1.3.4. Let $(A, \mathbf{B})$ be a hypergroup over $\mathbb{Z}$ and let

$$f : (A, \mathbf{B}) \longrightarrow (A, \mathbf{B})$$

be an automorphism of based rings. Then the anti-automorphism

$$\overline{f} : (A, \mathbf{B}) \longrightarrow (A, \mathbf{B})$$

is defined to be the composition of f with the anti-automorphism $\bar{\ }$ of axiom (H2).

Proposition 1.3.5. *Let $(A, \mathbf{B})$ be a hypergroup over $\mathbb{Z}$ and let*

$$f : (A, \mathbf{B}) \longrightarrow (A, \mathbf{B})$$

be an anti-automorphism of based rings. Then f is of the form $\overline{g}$ for a unique automorphism $g : (A, \mathbf{B}) \longrightarrow (A, \mathbf{B})$ of based rings.

Proof. Composing f with the hypergroup anti-automorphism $^{-}$, we obtain an automorphism g with the required properties. The uniqueness follows from the invertibility of $^{-}$. □

2. Tabular algebras

We now recall the definition of tabular algebras from [**13**].

2.1 Definition.

Definition 2.1.1. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. A *tabular algebra* is an $\mathcal{A}$-algebra A, together with a table datum $(\Lambda, \Gamma, B, M, C, *)$ where:

(A1) Λ is a finite poset. For each $\lambda \in \Lambda$, $(\Gamma(\lambda), B(\lambda))$ is a hypergroup over $\mathbb{Z}$ and $M(\lambda)$ is a finite set. The map

$$C : \coprod_{\lambda \in \Lambda} (M(\lambda) \times B(\lambda) \times M(\lambda)) \to A$$

is injective with image an $\mathcal{A}$-basis of A. We assume that $\mathrm{Im}(C)$ contains a set of mutually orthogonal idempotents $\{1_\varepsilon : \varepsilon \in \mathcal{E}\}$ such that $A = \sum_{\varepsilon, \varepsilon' \in \mathcal{E}} (1_\varepsilon A 1_{\varepsilon'})$ and such that for each $X \in \mathrm{Im}(C)$, we have $X = 1_\varepsilon X 1_{\varepsilon'}$ for some $\varepsilon, \varepsilon' \in \mathcal{E}$. A basis arising in this way is called a *tabular basis*.

(A2) If $\lambda \in \Lambda$, $S, T \in M(\lambda)$ and $b \in B(\lambda)$, we write $C(S, b, T) = C^b_{S,T} \in A$. Then $*$ is an $\mathcal{A}$-linear involutory anti-automorphism of A such that $(C^b_{S,T})^* = C^{\overline{b}}_{T,S}$, where $^{-}$ is the hypergroup anti-automorphism of $(\Gamma(\lambda), B(\lambda))$. If $g \in \mathbb{C}(v) \otimes_{\mathbb{Z}} \Gamma(\lambda)$ is such that $g = \sum_{b_i \in B(\lambda)} c_i b_i$ for some scalars c_i (possibly involving v), we write

$C^g_{S,T} \in \mathbb{C}(v) \otimes_{\mathcal{A}} A$ as shorthand for $\sum_{b_i \in B(\lambda)} c_i C^{b_i}_{S,T}$. We write $\mathbf{c}_\lambda$ for the image under C of $M(\lambda) \times B(\lambda) \times M(\lambda)$.

(A3) If $\lambda \in \Lambda$, $g \in \Gamma(\lambda)$ and $S, T \in M(\lambda)$ then for all $a \in A$ we have

$$a.C^g_{S,T} \equiv \sum_{S' \in M(\lambda)} C^{r_a(S',S)g}_{S',T} \mod A(< \lambda),$$

where $r_a(S', S) \in \Gamma(\lambda)[v, v^{-1}] = \mathcal{A} \otimes_{\mathbb{Z}} \Gamma(\lambda)$ is independent of T and of g and $A(< \lambda)$ is the $\mathcal{A}$-submodule of A generated by the set $\bigcup_{\mu<\lambda} \mathbf{c}_\mu$.

In all the examples we consider in this paper, the tabular basis will contain the identity element of the algebra. This means that the set $\mathcal{E}$ contains only the identity element of A.

The paper [**13**] also defines a more restrictive class of tabular algebras called "tabular algebras with trace". Since we are mainly concerned with representation theory and not Kazhdan–Lusztig theory in this paper, tabular algebras with trace will not be our primary objects of study. However, we recall the definition here for later reference. To do this, we need to recall the notion of $\mathbf{a}$-function, due to Lusztig.

Definition 2.1.2. Let $g_{X,Y,Z} \in \mathcal{A}$ be one of the structure constants for the tabular basis $\mathrm{Im}(C)$ of A, namely

$$XY = \sum_Z g_{X,Y,Z} Z,$$

where $X, Y, Z \in \mathrm{Im}(C)$. Define, for $Z \in \mathrm{Im}(C)$,

$$\mathbf{a}(Z) = \max_{X,Y \in \mathrm{Im}(C)} \deg(g_{X,Y,Z}),$$

where the degree of a Laurent polynomial is taken to be the highest power of v occurring with nonzero coefficient. We define $\gamma_{X,Y,Z} \in \mathbb{Z}$ to be the coefficient of $v^{\mathbf{a}(Z)}$ in $g_{X,Y,Z}$; this will be zero if the bound is not achieved.

Definition 2.1.3. A *tabular algebra with trace* is a tabular algebra in the sense of Definition 2.1.1 that satisfies the conditions (A4) and (A5) below.

(A4) Let $K = C^b_{S,T}$, $K' = C^{b'}_{U,V}$ and $K'' = C^{b''}_{X,Y}$ lie in $\mathrm{Im}(C)$. Then the maximum bound for $\deg(g_{K,K',K''})$ in Definition 2.1.2 is achieved if and only if $X = S$, $T = U$, $Y = V$ and b'' occurs with nonzero coefficient in bb'. If these conditions all hold and furthermore $b = b' = b'' = 1$, we require $\gamma_{K,K',K''} = 1$.

(A5) There exists an $\mathcal{A}$-linear function $\tau : A \longrightarrow \mathcal{A}$ (the *tabular trace*), such that $\tau(x) = \tau(x^*)$ for all $x \in A$ and $\tau(xy) = \tau(yx)$ for all $x, y \in A$, that has the property that for every $\lambda \in \Lambda$, $S, T \in M(\lambda)$, $b \in B(\lambda)$ and $X = C^b_{S,T}$, we have

$$\tau(v^{\mathbf{a}(X)}X) = \begin{cases} 1 \mod v^{-1}\mathcal{A}^- & \text{if } S = T \text{ and } b = 1, \\ 0 \mod v^{-1}\mathcal{A}^- & \text{otherwise.} \end{cases}$$

Here, $\mathcal{A}^- := \mathbb{Z}[v^{-1}]$.

The following notion is convenient for describing certain tabular algebras as subalgebras of other tabular algebras appearing in this paper.

Definition 2.1.4. Let A be a tabular algebra with table datum $(\Lambda, \Gamma, B, M, C, *)$. A *subdatum* of such a table datum is a tuple $(\Lambda', \Gamma', B', M', C', *')$ such that:

(S1) $(\Lambda', \leq')$ is a subposet of $(\Lambda, \leq)$;

(S2) for each $\lambda' \in \Lambda'$, $M'(\lambda')$ is a subset of $M(\lambda')$ and there is a unital monomorphism of based rings $(\Gamma'(\lambda'), B'(\lambda')) \longrightarrow (\Gamma(\lambda'), B(\lambda'))$ identifying $(\Gamma'(\lambda'), B'(\lambda'))$ with a subhypergroup of $(\Gamma(\lambda'), B(\lambda'))$;

(S3) under the above identifications, the maps C' and $*'$ are the restrictions of C and $*$, respectively, and $\mathrm{Im}(C') = A'$ is an $\mathcal{A}$-subalgebra of A.

The tuple defined above turns out to be a table datum for A', as we now show.

Proposition 2.1.5. *Let A be a tabular algebra with table datum $(\Lambda, \Gamma, B, M, C, *)$, and let $(\Lambda', \Gamma', B', M', C', *')$ be a subdatum for an $\mathcal{A}$-subalgebra A' of A. If the algebra A' contains all the idempotents $\{1_\varepsilon : \varepsilon \in \mathcal{E}\}$ of axiom (A1) then the given* *subdatum is a table datum for A'. If, furthermore, A is a tabular algebra with trace, then so is A'.*

Proof. We check the tabular axioms applied to A'. Axiom (A1) follows from the definitions and the hypothesis about the idempotents. Axiom (A2) follows from the definitions and Proposition 1.3.3. Axiom (A3) is immediate.

Suppose now that A is a tabular algebra with trace.

Let $X = C'^{b}_{S,T} \in \mathrm{Im}(C')$, where $S, T \in M'(\lambda')$ and $b \in B'(\lambda')$ for some $\lambda' \in \Lambda'$. We first show that the $\mathbf{a}$-functions $\mathbf{a}_{A'}$ and $\mathbf{a}_A$ arising from the algebras A' and A take the same value on X. It is clear from the definition of the $\mathbf{a}$-function and the fact that $\mathrm{Im}(C') \subseteq \mathrm{Im}(C)$ that $\mathbf{a}_{A'}(X) \leq \mathbf{a}_A(X)$. For the reverse inequality, we recall from [**13**, Lemma 2.2.3] that $C^b_{S,T}$ occurs in the product $C^1_{S,S} C^b_{S,T}$ with coefficient of degree $\mathbf{a}_A(C^b_{S,T})$. Now $C^b_{S,T} \in \mathrm{Im}(C')$ by definition of X, and $C^1_{S,S} \in \mathrm{Im}(C')$ because $S \in M'(\lambda')$ and $1 \in B'(\lambda')$ by the unital requirement of axiom (S2) in Definition 2.1.4. Thus X occurs in the product $C'^1_{S,S} C'^b_{S,T}$ with coefficient of degree $\mathbf{a}_A(X)$, and this implies that $\mathbf{a}_{A'}(X) \geq \mathbf{a}_A(X)$, as required.

Axiom (A4) follows from the aforementioned compatibility of $\mathbf{a}$-functions and the definitions, and axiom (A5) follows by restricting the trace τ of A to A'. □

2.2 The main result.

We are now ready to state our main result. Most of the rest of the paper will be devoted to studying examples of Theorem 2.2.1.

Theorem 2.2.1. *Let A be a tabular algebra of finite rank with table datum $(\Lambda, \Gamma, B, M, C, *)$; that is $|B(\lambda)| < \infty$ for each $\lambda \in \Lambda$.*

Let R be a commutative ring with identity. Suppose that α is an R-algebra automorphism of A satisfying $\alpha(\mathbf{B}) = \mathbf{B}$ and with the property that, for each $\lambda \in \Lambda$, there exists a permutation σ_λ of $M(\lambda)$ and an automorphism f_λ of the based ring $(\Gamma(\lambda), B(\lambda))$ such that

$$\alpha(C^b_{S,T}) = C^{f_\lambda(b)}_{\sigma_\lambda(S), \sigma_\lambda(T)}$$

for each $S, T \in M(\lambda)$ and $b \in B(\lambda)$.

Suppose that, for some $R \geq \mathbb{Z}$ and for each $\lambda \in \Lambda$, the algebra $R \otimes_{\mathbb{Z}} \Gamma(\lambda)$ is cellular over R with cell datum $(\Lambda_\lambda, M_\lambda, C_\lambda, \overline{f_\lambda})$, where $\overline{f_\lambda}$ is as in Definition 1.3.4.

Then $R \otimes_{\mathbb{Z}} A$ is cellular over $R \otimes_{\mathbb{Z}} \mathcal{A}$ with cell datum $(\Lambda', M', C', *')$, where $\Lambda' := \{(\lambda, \lambda') : \lambda \in \Lambda, \lambda' \in \Lambda_\lambda\}$ (ordered lexicographically), $M'((\lambda, \lambda')) := M(\lambda) \times M_\lambda(\lambda')$, $C'((S, s), (T, t))$ (where $(S, s), (T, t) \in M(\lambda) \times M_\lambda(\lambda')$) is equal to $C^{C_\lambda(s,t)}_{S,\sigma_\lambda(T)}$ and $*' = * \circ \alpha = \alpha \circ *$, so that $*' : C^b_{S,T} \mapsto C^{\overline{f_\lambda}(b)}_{\sigma_\lambda(T),\sigma_\lambda(S)}$.

Proof. Axiom (C1) for $R \otimes_{\mathbb{Z}} A$ follows from axiom (A1) applied to A and axiom (C1) applied to each hypergroup $(\Gamma(\lambda), B(\lambda))$.

We have $* \circ \alpha = \alpha \circ *$ by Proposition 1.3.3. It then follows from axiom (A2) that $*' = * \circ \alpha = \alpha \circ *$ is an anti-automorphism, and axiom (C2) follows because each hypergroup $R \otimes_{\mathbb{Z}} \Gamma(\lambda)$ is cellular with respect to the hypergroup anti-automorphism $\overline{f_\lambda}$.

To prove axiom (C3), let $\lambda \in \Lambda$ and let $C_\lambda(s, t)$ be a basis element of $\Gamma(\lambda)$ with $s, t \in M_\lambda(\lambda')$. Then by axiom (A3) we have, for any $a \in A$,

$$a.C^{C_\lambda(s,t)}_{S,\sigma_\lambda(T)} \equiv \sum_{S' \in M(\lambda)} C^{r_a(S',S)C_\lambda(s,t)}_{S',\sigma_\lambda(T)} \mod A(< \lambda).$$

Since $R \otimes_{\mathbb{Z}} \Gamma(\lambda)$ is cellular over R with cell basis given by C_λ, it follows by axiom (C3) applied to $R \otimes_{\mathbb{Z}} \Gamma(\lambda)$ that

$$r_a(S', S)C_\lambda(s, t) \equiv \sum_{s' \in M_\lambda(\lambda')} r'(S', S, s', s)C_\lambda(s', t) \mod R \otimes \Gamma_\lambda(< \lambda'),$$

where the $r'(S', S, s', s)$ are elements of $R \otimes_{\mathbb{Z}} \mathcal{A}$ that are independent of t (and, by axiom (A3), independent of $\sigma_\lambda(T)$). Axiom (C3) follows by tensoring over R. $\square$

Remark 2.2.2. In the special case where the automorphism α is the identity map, Theorem 2.2.1 reduces to [**13**, Theorem 2.1.1].

Remark 2.2.3. The theorem can be proved using a weaker order on Λ', namely the order such that $(\lambda_1, \lambda'_1) \leq (\lambda_2, \lambda'_2)$ if and only if $\lambda_1 \leq \lambda_2$ and $\lambda'_1 \leq \lambda'_2$. The proof is the same.

The next result shows that if α is not the identity map, then it must be an involution.

Lemma 2.2.4. *Let α be an automorphism satisfying the hypotheses of Theorem 2.2.1. Then α^2 is the identity map.*

Proof. This is immediate from the assumptions that $\alpha \circ *$ has order 2, $*$ has order 2, and α commutes with $*$. $\square$

It will also turn out that the map α in Theorem 2.2.1 need not be unique; see Remark 5.3.7 below.

Remark 2.2.5. The above results suggest a place to look for cellular involutions of a tabular algebra A in the case where the tabular involution does not work, namely to compose the tabular involution with basis-preserving algebra automorphisms of order 2.

3. Diagram algebra preliminaries

In the rest of the paper we study examples of cellular algebras with bases consisting of diagrams of various kinds. All of our examples can be related to each other. Although the ordinary Temperley–Lieb algebra (see §3.2) is arguably the hub of these connections, it is convenient to start by recalling Brauer's centralizer algebra. Some useful references on this algebra are Brauer's original paper [**6**], as well as [**10**, §4] and [**34**].

3.1 Brauer diagrams.

Combinatorially, the Brauer algebra B_n has defining basis consisting simply of the set of partitions of $2n$ objects into pairs. It is natural, however, to provide a graphical realisation. We start by recalling Jones' formalism of k-boxes [**19**]. For further details and references, the reader is referred to [**14**, §2].

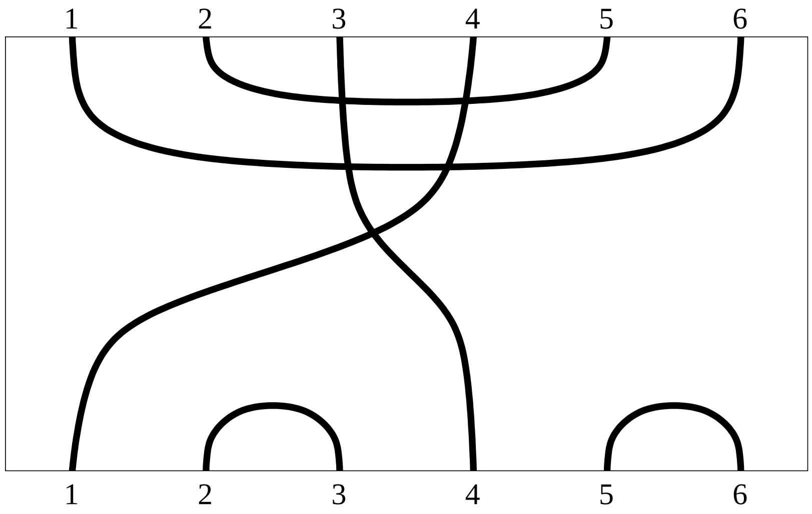
Definition 3.1.1. Let k be a nonnegative integer. The *standard k-box*, $\mathcal{B}_k$, is the set $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq k+1,\ 0 \leq y \leq 1\}$, together with the $2k$ marked points

$$1 = (1,1),\ 2 = (2,1),\ 3 = (3,1),\ \ldots,\ k = (k,1),$$
$$k+1 = (k,0),\ k+2 = (k-1,0),\ \ldots,\ 2k = (1,0).$$

(This is called the *Temperley–Lieb numbering*. The *Brauer numbering* renumbers the points $k+i$ of the standard k-box (for $1 \leq i \leq k$) as $k+1-i$. See Figure 1.)

Definition 3.1.2. Let X and Y be embeddings of some topological spaces (such as lines) into the standard k-box. Multiplication of such embeddings to obtain a new embedding in the standard k-box shall, where appropriate, be defined via the following procedure on k-boxes. The product XY is the embedding obtained by placing X on top of Y (that is, X is first shifted in the plane by $(0,1)$ relative to Y, so that marked point $(i,0)$ in X coincides with $(i,1)$ in Y), rescaling vertically by a scalar factor of $1/2$ and applying the appropriate translation to recover a standard k-box.

FIGURE 1. A Brauer algebra basis element for $n = 6$



Definition 3.1.3. Let k be a nonnegative integer. Consider the set of smooth embeddings of a single curve (which we usually call an "edge") in the standard k-box, such that the curve is either closed (isotopic to a circle) or its endpoints

coincide with two marked points of the box, with the curve meeting the boundary of the box only at such points, and there transversely.

By a smooth diffeomorphism of this curve we mean a smooth diffeomorphism of the copy of $\mathbb{R}^2$ in which it is embedded, that fixes the boundary, and in particular the marked points, of the k-box, and takes the curve to another such smooth embedding. (Thus, the orbit of smooth diffeomorphisms of one embedding contains all embeddings with the same endpoints.)

A concrete Brauer diagram is a set of such embedded curves with the property that every marked point coincides with an endpoint of precisely one curve. (In examples we can represent this set by drawing all the curves on one copy of the k-box. Examples can always be chosen in which no ambiguity arises thereby; see Figure 1.)

Two such concrete diagrams are said to be equivalent if one may be taken into the other by applying smooth diffeomorphisms to the individual curve embeddings within it.

There is an obvious map from the set of concrete diagrams to the set of pair partitions of the $2k$ marked points. It will be evident that the image under this map is an invariant of concrete diagram equivalence.

The set $B_k(\emptyset)$ is the set of equivalence classes of concrete diagrams. Such a class (or any representative) is called a Brauer diagram.

Let D_1, D_2 be concrete diagrams. Since the k-box multiplication defined above internalises marked points in coincident pairs, corresponding curve endpoints in $D_1 D_2$ may also be internalised seamlessly. Each chain of curves concatenated in this way may thus be put in natural correspondence with a single curve. Thus the multiplication gives rise to a closed associative binary operation on the set of concrete diagrams. It will be evident that this passes to a well defined multiplication on $B_k(\emptyset)$. Let R be a commutative ring with 1. The elements of $B_n(\emptyset)$ form the basis elements of an R-algebra $\mathcal{P}_n^B(\emptyset)$ with this multiplication.

A curve in a diagram that is not a closed loop is called *propagating* if its endpoints have different y-values, and *non-propagating* otherwise. (Some authors use the terms "through strings" and "arcs" respectively for curves of these types.)

Definition 3.1.4. The Brauer algebra $B_n = B_n(\delta)$ is the free $R[\delta]$-module with basis given by the elements of $B_n(\emptyset)$ with no closed loops. The multiplication is inherited from the multiplication on $\mathcal{P}_n^B(\emptyset)$ except that one multiplies by a factor of $\delta = v + v^{-1}$ for each resulting closed loop and then discards the loop.

In §5 we shall return to consider the tabularity of B_n and various related algebras.

In §4, we study examples of cellular algebras with basis diagrams consisting of non-intersecting curves that are inscribed in a k-box and labelled by elements of a certain ring. We call the associated algebras "decorated Temperley–Lieb algebras".

The original Temperley–Lieb algebras were defined by generators and relations, together with key representations, in [**33**]. They are quotients of the Hecke algebras associated to the symmetric groups. The Temperley–Lieb *diagram* algebra given in Definition 3.2.3 below is a realization of this algebra, meaning that it is isomorphic to the algebra of [**33**]. We will drop the word "diagram" for brevity.

3.2 Temperley–Lieb diagrams.

Note that in a Brauer diagram drawn on a single copy of the k-box it is not generally possible to keep the embedded curves disjoint (see Figure 1 for example). Let $T_k(\emptyset) \subset B_k(\emptyset)$ denote the subset of diagrams having representative elements in which the curves are disjoint. Representatives of this kind are called Temperley–Lieb diagrams.

It will be evident that $\mathcal{P}_n^B(\emptyset)$ has a subalgebra with basis the subset $T_k(\emptyset)$. (That is to say, the disjointness property is preserved under multiplication.) We denote this subalgebra $\mathcal{P}_n(\emptyset)$ (this may also be seen as a special case of [**19**, Definition 1.8]).

Because of the disjointness property there is, for each element of $T_k(\emptyset)$, a unique assignment of orientation to its curves that satisfies the following two conditions.

(i) A curve meeting the r-th marked point of the standard k-box, where r is odd, must exit the box at that point.

(ii) Each connected component of the complement of the union of the curves in the standard k-box may be oriented in such a way that the orientation of a curve coincides with the orientation induced as part of the boundary of the connected component.

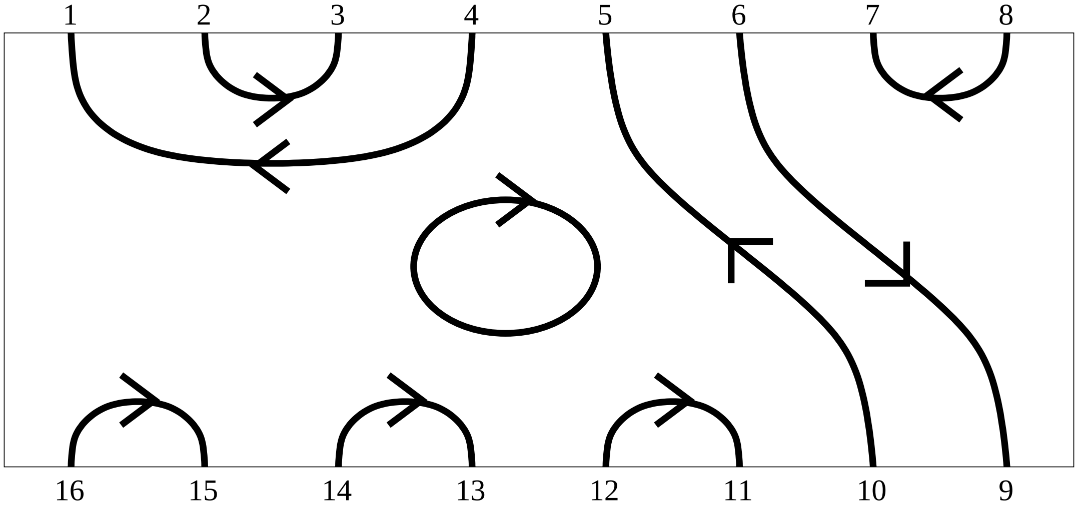
Note that the orientations match up automatically in composition.

Definition 3.2.1. [**7**] Given a diagram, a point x in the k-box is said to be l-exposed (to the western wall of the box) if l is the smallest number of edges it is necessary to traverse to get from x to the western wall. Again because of disjointness, every point on an edge has the same exposure. If this is l, then the edge is said to be l-exposed.

In composition each edge in the multiplied diagrams contributes a segment (possibly all) to an edge in the product, and it will be evident that the exposure of the new edge need not be the same as that of its ancestors. However the new exposure cannot exceed that of any ancestor.

Example 3.2.2. Let $k = 8$. An element of $T_8(\emptyset)$ is shown in Figure 2. Note that there are 10 connected components as in Definition 3.1.2 (ii), of which precisely 7 inherit a clockwise orientation.

FIGURE 2. Typical element of $T_8(\emptyset)$



Definition 3.2.3. Let R be a commutative ring with 1. The Temperley–Lieb algebra, $TL(n, \delta)$, is the free $R[\delta]$-module with basis given by the elements of $T_n(\emptyset)$ with no closed loops. The multiplication is inherited from the multiplication on $\mathcal{P}_n(\emptyset)$ except that one multiplies by a factor of δ for each resulting closed loop and then discards the loop.

We usually consider $TL(n, \delta)$ to be an algebra defined over $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$, where $\delta = v + v^{-1}$.

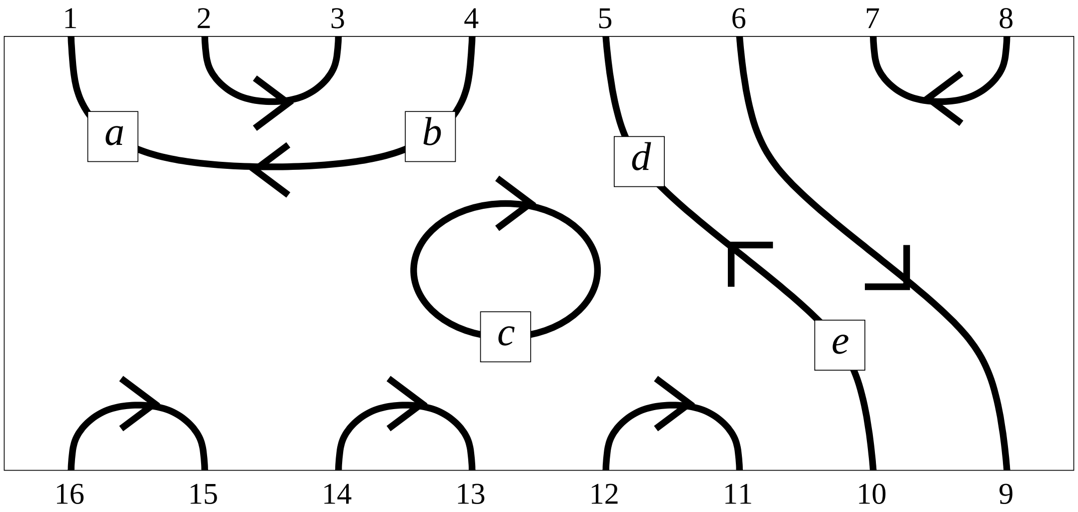
3.3 Decorated Temperley–Lieb algebras.

We now recall from [**19**, Example 2.2] the construction of the algebra P_n^A from the Temperley–Lieb algebra $TL(n, \delta)$ and the associative R-algebra A, where R is a commutative ring containing δ. The algebra A is assumed to have identity and a trace functional $\mathrm{tr} : A \longrightarrow R$ with $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ and $\mathrm{tr}(1) = \delta$.

Definition 3.3.1. Let A be as above, and let k be a nonnegative integer. We define the tangles $T_k(A)$ to be those that arise from elements of $T_k(\emptyset)$ by adding zero or more 1-boxes labelled by elements of A to each edge.

Figure 3 shows a typical element of $T_8(A)$ in which $a, b, c, d, e \in A$.

FIGURE 3. Typical element of $T_8(A)$



Definition 3.2.3 generalizes naturally to this situation, as follows.

Definition 3.3.2. Let k be a nonnegative integer and let A be an R-algebra (as before) with a free R-basis, $B_A = \{a_i : i \in I\}$, where $1 \in \{a_i\}$. The associative R-algebra P_k^A is the free R-module having as a basis those elements of $T_k(A)$ satisfying the conditions that

(i) all labels on edges are basis elements a_i,

(ii) each edge has precisely one label and

(iii) there are no closed loops.

The multiplication is defined on basis elements of P_k^A as above, and extended bilinearly. We start with the multiplication on $T_k(A)$, then apply relations (a), (b) and (c) below to express the product as an R-linear combination of basis elements, and finally, apply relation (d) below to remove any loops, multiplying by the scalar shown for each loop removed.

FIGURE 4. Relation (a) of Definition 3.3.2

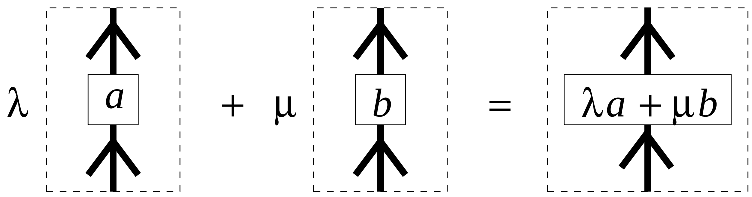


FIGURE 5. Relation (b) of Definition 3.3.2

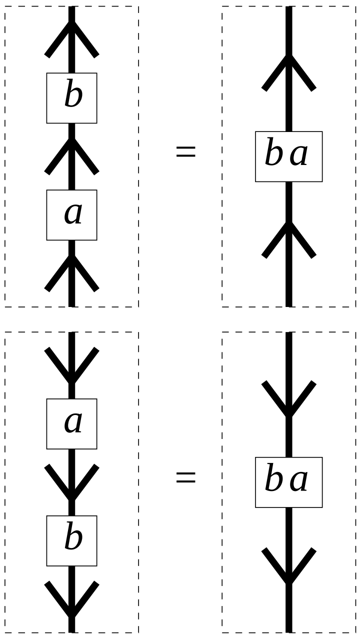


FIGURE 6. Relation (c) of Definition 3.3.2

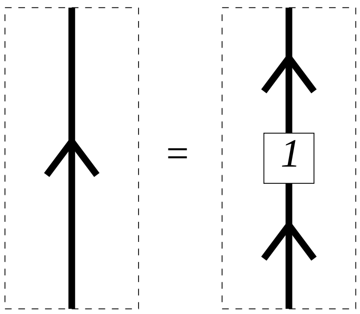
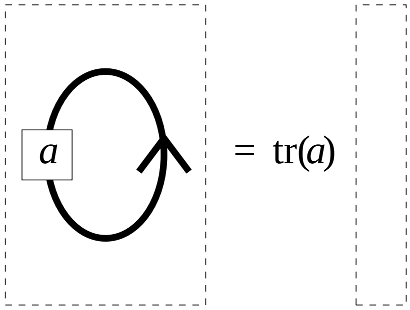


Figure 7. Relation (d) of Definition 3.3.2



We call the algebra P_k^A a *decorated Temperley–Lieb algebra* and the above basis, denoted $\mathbf{B}_n^A$, the *canonical basis* with respect to B_A.

For a proof that this procedure does define an associative algebra, the reader is referred to [**19**, Example 2.2].

Remark 3.3.3. The direction on the arrow in relation (d) is immaterial, although one can define a more intricate version of the algebra in which there are two traces on A, one for each orientation of the arrow.

As mentioned in [**19**], this construction might be regarded as a kind of wreath product of $TL(n)$ with A.

Fix a natural number l. Consider the subset of the canonical basis consisting of elements with the property that every edge with exposure greater than $l-1$ is decorated by the 1-box containing the identity element of A. It follows from the definition of exposure that this property is preserved under multiplication, and hence that the subset, denoted $\mathbf{B}_n^{A,l}$, forms a basis for a subalgebra $P_n^{A,l}$ of P_n^A.

For example, $P_n^{A,0}$ is isomorphic to the ordinary Temperley–Lieb algebra, while $P_n^{A,n} \cong P_n^A$. The algebras $P_n^{A,l}$ come from the contour algebra formalism introduced in [**7**].

Proposition 3.3.4. *Let A be a hypergroup over R with distinguished basis B and any trace map. There is an isomorphism ρ of R-algebras from $(A^{\otimes n}, B^{\otimes n})$ to the subalgebra of P_n^A spanned by all canonical basis elements with no non-propagating edges. The isomorphism takes basis elements to basis elements.*

Proof. Let $b = b_{i_1} \otimes b_{i_2} \otimes \cdots \otimes b_{i_n}$ be a typical basis element from the set $B^{\otimes n}$. This element is sent by the isomorphism, ρ, to a canonical basis element of P_n^A with no non-propagating edges, where the decoration on the k-th propagating edge (counting from 1 to n, starting at the left) is b_{i_k} if k is odd, and $b_{\overline{i_k}}$ if k is even.

For the proof that this construction defines an isomorphism of algebras, the reader is referred to [**14**, Proposition 2.3.4]. Note that the trace map plays no role in the structure of the algebra, because closed loops cannot arise. $\square$

Lemma 3.3.5 [**14**, Lemma 2.3.2]**.** *Let A be a hypergroup over $\mathbb{Z}$ with distinguished basis B and the trace map $\delta.t$, where t is as in Proposition 1.2.4. There is an linear anti-automorphism, $*$, of P_n^A permuting the canonical basis. The image, b^*, of a basis element b under this map is obtained by reflecting b in the line $y = 1/2$, reversing the direction of all the arrows and replacing each 1-box labelled by $b_i \in B$ by a 1-box labelled by $b_{\overline{i}}$.* $\square$

4. Tabularity of decorated Temperley–Lieb algebras and related algebras

4.1 General results.

The following result is [**14**, Theorem 3.2.3]; we recall part of the proof below as we require the construction later.

Theorem 4.1.1 [**14**]**.** *Let A be a hypergroup over $\mathbb{Z}$ with distinguished basis B and the trace map $\delta.t$, where t is as in Proposition 1.2.4. Then the algebra P_n^A equipped with its canonical basis $\mathbf{B}_n^A$ is a tabular algebra.*

Proof. We require to construct a table datum. Let Λ be the set of integers r with $0 \leq r \leq n$ and $n - r$ even, ordered in the usual way.

For $\lambda \in \Lambda$, let $(\Gamma(\lambda), B(\lambda))$ be the λ-th tensor power of the hypergroup (A, B) with the basis and anti-automorphism induced by Proposition 1.2.5.

Let $M(\lambda)$ be the set of possible configurations of $(n - \lambda)/2$ non-propagating edges with endpoints on the line $y = 1$ that arise from an element of $\mathbf{B}_n^A$. Let

$b = b_{i_1} \otimes b_{i_2} \otimes \cdots \otimes b_{i_\lambda}$ be a basis element of $B(\lambda)$ and let m and m' be elements of $M(\lambda)$. The map C produces a basis element in $\mathbf{B}_n^A$ from the triple (m, b, m') as follows. Turn the half-diagram corresponding to m' upside down, reverse the directions of all the arrows and relabel all 1-boxes labelled by $b_i \in B$ so they are labelled by $b_{\bar{i}}$. Join any free marked points in the line $y = 0$ to free marked points in the line $y = 1$ so that they do not intersect. Orient any new edges according to the orientation of the standard n-box. Decorate the λ propagating edges with the basis element b using the construction of Proposition 3.3.4. (See [**14**, Example 3.2.4] for an illustration of this.)

The map $*$ is the one given by Lemma 3.3.5.

For the proof that this construction defines a table datum, the reader is referred to [**14**, Theorem 3.2.3], which proves the stronger result that the construction defines a tabular algebra with trace. $\square$

Similarly, using the subdatum idea we have

Theorem 4.1.2. *Let A be a hypergroup over $\mathbb{Z}$ with distinguished basis B and the trace map $\delta.t$, where t is as in Proposition 1.2.4. Then the algebra $P_n^{A,l}$ equipped with its canonical basis $\mathbf{B}_n^{A,l}$ is a tabular algebra (for any appropriate l).*

Proof. The proof goes through as before (noting that the hypergroups appearing in the table datum are the λ-th tensor power of (A, B) for $\lambda < l$, and the l-th tensor power thereafter). $\square$

Theorem 4.1.3. *Suppose that R is a commutative ring with identity and (A, B) is a hypergroup such that $R \otimes_{\mathbb{Z}} A$ is cellular with respect to an anti-automorphism of the based ring A. Equip A with the trace map $\delta.t$, where t is the trace of Proposition 1.2.4. Then $R \otimes_{\mathbb{Z}} P_n^A$ is cellular, and an explicit cell datum is given by Theorem 2.2.1.*

Proof. By Proposition 1.3.5, the anti-isomorphism of the statement is of the form $\bar{g}$ for some automorphism g of based rings. The map g then induces a permutation

α of the basis diagrams of P_n^A by acting simultaneously on the decorations of each edge; note that α preserves the number of propagating edges in each diagram. By Proposition 1.2.4, we see that $t(g(x)) = t(x)$, because the action of g on an element $x \in A$ does not alter the coefficient of the identity element of $\mathbf{B}$. The relations of Definition 3.3.2 now show that α induces a basis-preserving automorphism of the algebra $R \otimes_{\mathbb{Z}} P_n^A$.

As explained in Theorem 4.1.1, the hypergroups occurring in the table datum for P_n^A are certain tensor powers of the hypergroup $(A, \mathbf{B})$ (see Proposition 1.2.5). The automorphism α induces the basis-preserving automorphism g^r on the hypergroup $(A^{\otimes r}, \mathbf{B}^r)$, and the latter hypergroup is cellular over R with respect to the anti-automorphism $\overline{g^r}$ by Example 1.1.6.

The result now follows by Theorem 2.2.1, in which α is as above and f_r is given by g^r. $\square$

4.2 Cyclotomic Temperley–Lieb algebras.

The so-called cyclotomic Temperley–Lieb algebras $TL_{n,m}(\delta_0, \delta_1, \ldots \delta_{m-1})$ are algebras over a ring R containing elements $\delta_0, \delta_1, \ldots, \delta_{m-1}$. They were introduced by Rui and Xi [**30**], both in terms of generators and relations [**30**, Definition 2.1], and equivalently in terms of a calculus of diagrams [**30**, Definition 3.3, Theorem 3.4].

The rules for manipulating the decorations on edges in [**30**] are somewhat intricate (see [**30**, §3]). We show here, for certain values of the parameters, how the algebra may also be set up using planar algebras on 1-boxes.

Remark 4.2.1. This construction is mentioned in passing by Cox et al [**7**, Remark 2.3], who then provide an entirely straightforward construction for an isomorphic diagram algebra and a sequence of generalizations, which we will not need here. It is also possible to define the algebras for general parameter values using planar algebras on 1-boxes, but this requires the use of two traces (see Remark 3.3.3) and we will not pursue this here.

Lemma 4.2.2. *Let (A, B) be the cyclic group $\mathbb{Z}_m$ considered as a hypergroup $(\mathbb{Z}\mathbb{Z}_m, \mathbb{Z}_m)$, equipped with the trace map $\delta.t$, where t is as in Proposition 1.2.4. Then P_n^A is isomorphic to the cyclotomic Temperley–Lieb algebra $TL_{n,m}(\delta, 0, 0, \dots, 0)$ over $\mathcal{A}$, and the canonical basis of P_n^A can be chosen to map to the diagram basis of $TL_{n,m}$.*

Proof. For $1 \leq k < n$, let $E_{k,n}$ be an element of $T_n(\emptyset)$ with no closed loops in which each point i is connected by a propagating edge to point $2n+1-i$, unless $i \in \{k, k+1, 2n-k, 2n+1-k\}$; furthermore, points k and $k+1$ are connected by a non-propagating edge, as are points $2n-k$ and $2n+1-k$.

Let g be a generator of the group $\mathbb{Z}_m$. For $1 \leq k \leq n$, let $T_{k,n}$ be the basis diagram corresponding to

$$\underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{k-1} \otimes g \otimes \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{n-k}$$

under the isomorphism of Proposition 3.3.4.

The map sending $E_{i,n}$ and $T_{j,n}$ to the respective elements E_i and T_j in $TL_{n,m}$ (see the notation of the proof of [**30**, Theorem 3.4]) extends uniquely to an isomorphism of $\mathcal{A}$-algebras. This is a matter of checking that the multiplications in the two diagram calculi are compatible, and this follows from the rules given in [**30**, §3]. □

The remarks in the proof of Theorem 4.1.1 now give

Corollary 4.2.3. *The cyclotomic Temperley–Lieb algebra $TL_{n,m}(\delta, 0, \dots, 0)$ is a tabular algebra with trace.* □

In contrast, for general parameter values, the cyclotomic Temperley–Lieb algebra is not tabular in any obvious way. This is not so surprising given that it is a multiparameter algebra, but even if we require all the parameters to lie in $\mathcal{A}$, complications arise. If $\delta_i = \delta_{m-i} \in \mathcal{A}$ for all $0 \leq i \leq m$, there is an $\mathcal{A}$-linear anti-automorphism of the algebra fixing the generators e_i and sending each t_j to t_j^{-1}, and this serves as a tabular anti-automorphism. In general, however, this map fails to be an $\mathcal{A}$-linear anti-automorphism, even if all the δ_i lie in $\mathcal{A}$. If one is primarily interested in the cellular structure, this is not a major problem, as we will explain in Remark 4.2.5.

Corollary 4.2.4 (Rui–Xi, [30, Theorem 5.3]). *Let R be a commutative ring with identity such that $x^m - 1$ splits into linear factors over $R[x]$. Then the cyclotomic Temperley–Lieb algebra $TL_{n,m}(\delta, 0, 0, \dots, 0)$ over R is cellular with respect to the map $* \circ \alpha = \alpha \circ *$, where $*$ is as defined in Lemma 3.3.5, and α is the map induced by applying the inversion automorphism of $\mathbb{Z}_m$ to each edge of each basis diagram.*

Proof. Let $P(x) = x^m - 1 \in R[x]$. By hypothesis, $P(x)$ splits into linear factors

$$P(x) = \prod_{i=1}^{k} (x - \lambda_i).$$

By Example 1.1.4, this shows that $R \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{Z}_m = R[x]/\langle P(x) \rangle$ is cellular with respect to the identity map. Since the identity map $\mathbb{Z}_m \longrightarrow \mathbb{Z}_m$ is an anti-automorphism of based rings, Theorem 4.1.3 and Lemma 4.2.2 show that $TL_{n,m}(\delta, 0, 0, \dots, 0)$ is cellular, and Theorem 2.2.1 provides a cell datum with the required properties. □

Remark 4.2.5. Rui and Xi [**30**, Theorem 5.3] prove the result above for arbitrary values of the parameters δ_i, but the cell datum remains essentially the same in each case. In particular, the map $* \circ \alpha = \alpha \circ *$ remains as the cellular anti-automorphism of $TL_{n,m}$ for all parameter values, even though $*$ (respectively, α) is not an anti-automorphism (respectively, an automorphism) of the algebra in general.

4.3 Other similar examples, and subdata.

Other algebras that can be treated similarly include the blob algebra and the generalized Temperley–Lieb algebra of type H_n. We will give only a sketch of the arguments, for the sake of brevity.

The blob algebra was defined by the second author and Saleur [**29**] in a statistical mechanical context. It may be defined as a certain subalgebra of P_n^A, where A is the algebra $\mathbb{Z}[x]/\langle x^2 - x \rangle$. Since A has no obvious hypergroup structure, we make the change of variables $y = 2x - 1$ and consider $A' = \mathbb{Z}[y]\langle y^2 - 1\rangle$; this is the hypergroup $(\mathbb{Z}\mathbb{Z}_2, \mathbb{Z}_2)$. (Note that $\mathbb{Q} \otimes_{\mathbb{Z}} A \cong \mathbb{Q} \otimes_{\mathbb{Z}} A'$; this has the effect of making the decoration in [**29**] unipotent instead of idempotent.) The blob algebra can be constructed from $P_n^{A'}$ using a subdatum to ensure that only edges exposed to the leftmost edge of the n-box may carry decorations; this means that most of the hypergroups appearing in the table datum are isomorphic to A'. Since A' is cellular with respect to the identity map (i.e., inversion in $\mathbb{Z}_2$), Theorem 2.2.1 can be applied with α taken to be the identity automorphism.

The generalized Temperley–Lieb algebras of type H_n were considered by Graham [**9**] and an explicit cell datum was constructed in [**12**, Theorem 3.3.5] in terms of a basis of diagrams that were later shown to be a tabular basis [**13**, Theorem 5.2.5]. The treatment of these algebras using Theorem 2.2.1 is similar to that of the blob algebra above (see also the remarks in [**13**, §2.1]). The main differences are (a) the relevant hypergroup to use is $A = \Gamma_H = \mathbb{Z}[x]/\langle x^2 - x - 1\rangle$ with basis B_H given by the images of 1 and x, and (b) more care is needed in defining the subdatum, which may be constructed using the rules for "H-admissibility" listed in [**14**, Definition 4.2.3].

5. Tabularity of the Brauer algebra and related algebras

5.1 The Brauer algebra.

We now recall the tabular structure of the Brauer algebra. This example comes from [**13**, Example 2.1.2] and [**15**, §4.2].

As in [**10**, §4], we may describe the basis diagrams in terms of certain triples.

Definition 5.1.1. Fix a diagram D. The integer $t(D)$ is defined to be the number of propagating edges. The involutions $S_1(D)$, $S_2(D)$ in the symmetric group $\mathcal{S}(n)$ are defined such that $S_i(D)$ interchanges the ends of the joins between points with the same y-coordinate. For example, with D as in Figure 1, we have $S_1(D) = (16)(25)$. Corresponding to these we have subsets $\mathrm{Fix}(S_i(D))$ of $\{1, \ldots, n\}$, which are the fixed points of the involutions $S_i(D)$. Finally, we have a permutation $w(D)$ in $\mathcal{S}(t)$, where $t = t(D)$; this is the permutation of $\mathrm{Fix}(S_1(D))$ determined by taking the end points of the propagating edges (regarded as joining from the row $y = 0$ to the row $y = 1$) in the order determined by taking their starting points in the row $y = 0$ in increasing order. (We consider $\mathcal{S}(0)$ to be the trivial group, in which case w is the identity.) The diagram D is then determined by the triple $[S_1(D), S_2(D), w(D)]$.

A table datum for the Brauer algebra (equipped with the diagram basis) may be given as follows. (This gives the algebra the structure of a tabular algebra with trace.)

Definition 5.1.2. Let B_n be the Brauer algebra (over $\mathcal{A}$) on n strings. The algebra has a table datum $(\Lambda, \Gamma, B, M, C, *)$ as follows.

Take Λ to be the set of integers i between 0 and n such that $n-i$ is even, ordered in the natural way. If $\lambda = 0$, take $(\Gamma(\lambda), B(\lambda))$ to be the trivial hypergroup; otherwise, take $\Gamma(\lambda)$ to be the group ring $\mathbb{Z}\mathcal{S}(\lambda)$ with basis $B(\lambda) = \mathcal{S}(\lambda)$ and involution $\overline{w} = w^{-1}$. Take $M(\lambda)$ to be the set of involutions on n letters with λ fixed points. Take $C(S_1, w, S_2) = [S_1, S_2, w]$; $\mathrm{Im}(C)$ contains the identity element. The anti-automorphism $*$ sends $[S_1, S_2, w]$ to $[S_2, S_1, w^{-1}]$.

We state the next result for later use.

Lemma 5.1.3.

(i) *The operation of reflecting each basis diagram of B_n in a vertical line $x = (n + 1)/2$ extends to a unique automorphism of $\mathcal{A}$-algebras, ρ, of B_n.*

(ii) *Let $\omega_k : \mathcal{S}_k \longrightarrow \mathcal{S}_k$ be the automorphism of the symmetric group obtained by conjugation by the longest element of $\mathcal{S}_k$. Then we have*

$$\rho([S_1, S_2, w]) = [\omega_n(S_1), \omega_n(S_2), \omega_t(w)],$$

where t is the number of fixed points of S_1 and S_2.

Proof. Part (i) follows easily from the definition of the multiplication in B_n.

Part (ii) is a consequence of the observation that if $g \in \mathcal{S}_k$, we have $g(i) = j$ if and only if $(\omega_k(g))(n+1-i) = n+1-j$. □

Remark 5.1.4. As mentioned in [**13**, §2.1], similar techniques may be applied to the case of the partition algebra of [**27**]; again the hypergroups are symmetric groups equipped with inversion as the involution. This recovers Xi's main result in [**35**].

5.2 The walled Brauer algebra.

The walled Brauer algebra, also known as the rational Brauer algebra, is a certain subalgebra of the Brauer algebra first considered by Benkart et al in [**5**, §5]. The cellularity of this algebra is well-known to the experts, and it is implicit in the construction of the basis described in [**22**]. We include the example here to illustrate how easy it is to describe the cellular structure of this algebra using our techniques.

Definition 5.2.1. Let $(\Lambda, \Gamma, B, M, C, *)$ be the table datum for the Brauer algebra B_n given in Definition 5.1.2, let r and s be positive integers with $r + s = n$, and define $(\Lambda', \Gamma', B', M', C', *')$, as follows. Let

$$\Lambda' = \{\lambda' \in \Lambda : \lambda' \geq |r - s|\},$$

and for each $\lambda' \in \Lambda'$, define $(\Gamma'(\lambda'), B'(\lambda'))$ to be the hypergroup corresponding to the subgroup $\mathcal{S}(r') \times \mathcal{S}(s')$ of $\mathcal{S}(\lambda')$, where r' and s' are the unique nonnegative integers satisfying $r' + s' = \lambda'$ and $r' - s' = r - s$. For $\lambda' \in \Lambda'$ and the corresponding integers r' and s' just given, denote by P^- the set $\{1, 2, \ldots, r\}$ and by P^+ the set $\{r+1, r+2, \ldots, n\}$. We define $M'(\lambda')$ to be the subset of $M(\lambda')$ consisting of involutions S for which the following conditions hold:

(i) S has r' fixed points in P^-;
(ii) S has s' fixed points in P^+;
(iii) if S exchanges two distinct points then one of the points comes from P^- and the other from P^+.

We define C' and $*'$ to be the restrictions of C and $*$ to the appropriate domains.

Definition/Lemma 5.2.2. *The tuple* $(\Lambda', \Gamma', B', M', C', *')$ *is a subdatum for the table datum of the Brauer algebra, and corresponds to an algebra* A'*, which is by definition the walled Brauer algebra,* $B_{r,s}$*.*

Proof. The walled Brauer algebra $B_{r,s}$ defined in [**5**, §5] is given to be that spanned by certain basis diagrams, called (r, s)-diagrams. A routine check shows that the (r, s)-diagrams are precisely those in the image of the map C'. It remains to be checked that $\mathrm{Im}(C')$ is a subalgebra of A, but this also presents no difficulties (see [**5**, §5]). □

Corollary 5.2.3. *The walled Brauer algebra* $B_{r,s}$ *is cellular with respect to the anti-automorphism* $*'$ *of Lemma 5.2.2.*

Proof. Since $B_{r,s}$ contains the identity element of the Brauer algebra B_{r+s}, Proposition 2.1.5 and Lemma 5.2.2 show that $(\Lambda', \Gamma', B', M', C', *')$ is a table datum for $B_{r,s}$ (and furthermore, that $B_{r,s}$ is a tabular algebra with trace).

Example 1.1.3 shows that $\mathbb{Z}\mathcal{S}_n$ is cellular over $\mathbb{Z}$ with respect to the group inversion. Example 1.1.6 then shows that $\mathbb{Z}\mathcal{S}_r \otimes_{\mathbb{Z}} \mathbb{Z}\mathcal{S}_s \cong \mathbb{Z}(\mathcal{S}_r \times \mathcal{S}_s)$ is cellular with respect to inversion in the group $\mathcal{S}_r \times \mathcal{S}_s$.

Using the above observations, Theorem 2.2.1 (with α as the identity map) now constructs a cell datum showing that $B_{r,s}$ is cellular with respect to $*' \circ \alpha = *'$, as required. □

Remark 5.2.4. Notice that Theorem 5.2.3 includes the cellularity of the Brauer algebra as a special case; this was originally a theorem of Graham–Lehrer [**10**, Theorem 4.10].

5.3 Jones' annular algebra.

Jones' annular algebra (or the Jones algebra, for short) is a certain subalgebra of the Brauer algebra that is also a quotient of an affine Hecke algebra of type A. It was introduced in [**18**] and first shown to be cellular in [**10**, §6]. In order to define the algebra, we recall the notion of an annular involution.

Definition 5.3.1. An involution $S \in \mathcal{S}_n$ is annular if and only if for each pair i, j interchanged by S (with $(i < j)$) and $P_{i,j} = \{k : i \leq k \leq j\}$, we have

(a) $S(P_{i,j}) = P_{i,j}$ and

(b) either S fixes no element of $P_{i,j}$ or every element fixed by S is contained in $P_{i,j}$.

Definition 5.3.2. Let $(\Lambda, \Gamma, B, M, C, *)$ be the table datum for the Brauer algebra given by Definition 5.1.2. We define $\Lambda' = \Lambda$, and for each $\lambda' \in \Lambda$, let $(\Gamma(\lambda'), B(\lambda'))$ be the cyclic group of order λ' considered as a hypergroup, unless $\lambda' = 0$, in which case we define $(\Gamma(\lambda'), B(\lambda'))$ to be the trivial hypergroup. Also for each $\lambda' \in \Lambda'$, we define $M(\lambda')$ to be the set of annular involutions with λ' fixed points; it may be checked that this is always a nonempty set. We define C' and $*'$ to be the restrictions of C and $*$ to the appropriate domains.

Definition/Lemma 5.3.3. *The tuple $(\Lambda', \Gamma', B', M', C', *')$ is a subdatum for the table datum of the Brauer algebra, and corresponds to an algebra A', which is by definition Jones' annular algebra, J_n.*

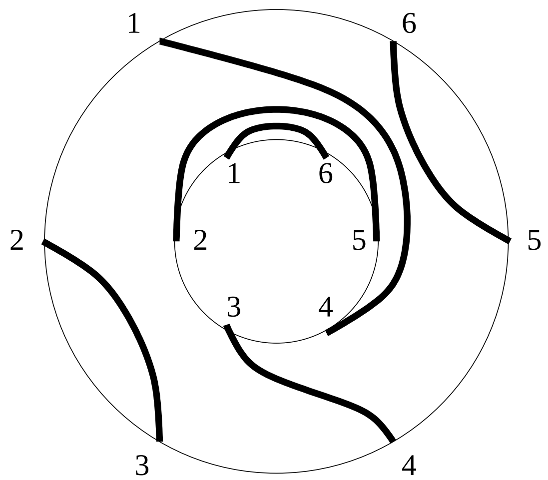
Proof. The definition of J_n given in [**10**, §6] is as the span of those Brauer algebra diagrams $[S_1, S_2, w]$, where S_1 and S_2 are annular involutions with t fixed points, and where w is an element of the cyclic group of order t if $t > 0$, with $w = 1$ if $t = 0$. It is not hard to see that this agrees with our construction. It is also routine to check that this defines a subalgebra; see [**18**] or [**10**, §6]. □

Note that the tabular structure of the Jones algebra has already been described in [**13**, Example 2.1.4].

Example 5.3.4. Let $[S_1, S_2, w]$ be the basis diagram shown in Figure 1. The involution S_1 is annular because the subsets $\{k : 2 \leq k \leq 5\}$ and $\{k : 1 \leq k \leq 6\}$ contain all the fixed points of S_1. The involution S_2 is annular because S_2 fixes no elements in the sets $\{k : 2 \leq k \leq 3\}$ and $\{k : 5 \leq k \leq 6\}$. Note that w lies in the cyclic group of order 2, the number of fixed points of each of S_1 and S_2.

The reason for the term "annular" is that the basis diagrams of J_n are precisely those that can be inscribed without intersections within an annulus. Although the diagram in Figure 1 has intersections when inscribed in a rectangle, it can be inscribed without intersections in an annulus, as shown in Figure 8.

FIGURE 8. A basis element of Jones' annular algebra



A cell datum for the Jones algebra cannot be obtained by restriction of the cell datum for the Brauer algebra, the obstruction being essentially that group algebras of cyclic groups are not usually cellular with respect to the group inversion. However, we can use our main results to exploit the fact that these group algebras are cellular with respect to the identity anti-automorphism (see Example 1.1.4 and the proof of Corollary 4.2.4).

Lemma 5.3.5. *The automorphism ρ of the Brauer algebra B_n, defined in Lemma 5.1.3, restricts to an automorphism of J_n.*

Proof. If S_1 is an annular involution, it follows from the symmetric nature of Definition 5.3.1 that $\omega_n(S_1)$ is also annular. If $w \in \mathbb{Z}_t$, we find that $\omega_t(w) = w^{-1}$. The result now follows from Lemma 5.1.3 (ii). □

Corollary 5.3.6 (Graham–Lehrer, [10, Theorem 6.15]). *Let $n \in \mathbb{N}$, and let R be a commutative ring with identity such that $x^t - 1$ splits into linear factors over $R[x]$ for all $0 \leq t \leq n$ such that $n - t$ is even. Then the Jones algebra J_n over*

$R[v, v^{-1}]$ is cellular with respect to the map $* \circ \rho = \rho \circ *$, where $*$ is the map of Definition 5.1.2, and ρ is the map of Lemma 5.3.5.

Proof. Using Example 1.1.4 as in the proof of Corollary 4.2.4, we see that $R\mathbb{Z}_t$ is cellular over R with respect to the identity map, for all values of t given in the statement. We then apply Theorem 2.2.1 with ρ in the role of α. Note that in this case, the maps f_λ all arise from inversion in suitable cyclic groups, and the maps σ_λ are the maps ω_n from Lemma 5.1.3. The composite maps $\overline{f_\lambda}$ are all equal to identity maps. Theorem 2.2.1 completes the proof by constructing a cell datum. □

Remark 5.3.7. It is possible to exploit the rotational symmetry in the Jones algebra to define automorphisms of J_n other than ρ for which Corollary 5.3.6 still holds. This shows that the automorphism α required by Theorem 2.2.1 need not be uniquely determined.

Remark 5.3.8. Our approach in this paper provides a convenient framework for examining situations such as the embedding of the Jones algebra in the Brauer algebra, where the cellular structures are not compatible, but the tabular structures are.

Acknowledgements

The first author thanks Stephen Doty and Steffen Koenig for helpful conversations and correspondence.

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