COMPLETIONS OF CELLULAR ALGEBRAS

R.M. Green

Department of Mathematics and Statistics
Lancaster University
Lancaster LA1 4YF
England
E-mail: r.m.green@lancaster.ac.uk

Abstract. We introduce procellular algebras, so called because they are inverse limits of finite dimensional cellular algebras as defined by Graham and Lehrer. A procellular algebra is defined as a certain completion of an infinite dimensional cellular algebra whose cell datum is of "profinite type". We show how these notions overcome some known obstructions to the theory of cellular algebras in infinite dimensions.

To appear in Communications in Algebra

0. Introduction

Cellular algebras, which were introduced by Graham and Lehrer in [**8**], are a class of associative algebras defined in terms of a "cell datum" satisfying certain axioms. Several interesting classes of finite dimensional algebras, including certain Hecke algebras and Brauer algebras, can be described in this way.

The author was supported in part by an E.P.S.R.C. postdoctoral research assistantship.

1991 Mathematics Subject Classification: 22A30.

One of the strengths of the theory of cellular algebras is that it provides a complete list of absolutely irreducible modules for the algebra over a field. Unfortunately, this property is no longer true if one allows an algebra satisfying the cellular axioms to be infinite dimensional, and there is a simple counterexample to show this.

The goal of this paper is to give a natural definition of cellularity for a general algebra which not only agrees with the original definition if the algebra is finite dimensional, but which also produces results analogous to those in the finite dimensional case if the algebra is infinite dimensional.

The material in §1 is expository, giving a summary of some of the important results in the theory of cellular algebras, and outlining the obstructions to the theory in infinite dimensions. The main problem is that although the definition of "cellular" makes sense for infinite as well as finite dimensional algebras, the results of [**8**] assume tacitly that the algebra is finite dimensional. If the algebra is infinite dimensional, the results can fail badly even for simple examples.

In §2, we introduce the notion of a cell datum of "profinite type". For such a cell datum, the associated partially ordered set is allowed to be infinite, provided that it satisfies a certain finitary condition. If the condition is satisfied, the infinite dimensional algebra of profinite type may be completed with respect to a natural topology, which depends on the cell datum, resulting in a "procellular" algebra. The desired classification for absolutely irreducible modules can then be obtained by considering "smooth" representations of the completed algebra.

In §3, we look at a nontrivial natural example of such a procellular algebra. This example arises as a completion of Lusztig's algebra $\dot{U}$, which appears in [**13**], or alternatively as a projective limit of q-Schur algebras. We also discuss briefly another very interesting natural example of a procellular algebra which arises from a certain completion of the affine Temperley–Lieb algebra.

1. Cellular algebras

1.1 Definitions.

We start by reviewing some of the basic properties of cellular algebras.

Definition 1.1.1 ([8, Definition 1.1]). *Let R be a commutative ring with identity. A cellular algebra over R is an associative unital algebra, A, together with a cell datum $(\Lambda, M, C, *)$ where*

1. *Λ is a partially ordered set (or "poset" for short), ordered by $<$. For each $\lambda \in \Lambda$, $M(\lambda)$ is a finite set (the set of "tableaux" of type λ) such that*
$$C : \coprod_{\lambda \in \Lambda} (M(\lambda) \times M(\lambda)) \to A$$
is injective with image an R-basis of A.
2. *If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, we write $C(S,T) = C^\lambda_{S,T} \in A$. Then $*$ is an R-linear involutory anti-automorphism of A such that $(C^\lambda_{S,T})^* = C^\lambda_{T,S}$.*
3. *If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ then for all $a \in A$ we have*
$$aC^\lambda_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^\lambda_{S',T} \mod A(< \lambda),$$
where $r_a(S', S) \in R$ is independent of T and $A(< \lambda)$ is the R-submodule of A generated by the set
$$\{C^\mu_{S'',T''} : \mu < \lambda, S'' \in M(\mu), T'' \in M(\mu)\}.$$

We use Definition 1.1.1 only when Λ is a finite set, although such an assumption is omitted in [**8**].

There are many interesting examples of finite dimensional algebras which satisfy these axioms, some of which are described in detail in [**8**]. These include the Brauer algebra, the Temperley–Lieb algebra, the Hecke algebra of type A (using the Kazhdan–Lusztig basis as the cell basis) and Jones' annular algebra.

The theory of cellular algebras may be applied to find, among other things, criteria for semisimplicity and a complete set of absolutely irreducible modules over a field. In this paper, we will mostly be concerned with the latter application, and to understand it, we recall the notion of cell modules.

Definition 1.1.2 ([8, Definition 2.1]. *For each $\lambda \in \Lambda$ define the left A-module $W(\lambda)$ as follows: $W(\lambda)$ is a free R-module with basis $\{C_S : S \in M(\lambda)\}$ and A-action defined by*

$$aC_S = \sum_{S' \in M(\lambda)} r_a(S', S) C_{S'} \quad (a \in A, S \in M(\lambda)),$$

where the r_a are the structure constants of Definition 1.1.1. This is called the cell representation of A corresponding to $\lambda \in \Lambda$.

Associated with each cell representation $W(\lambda)$ is a bilinear form ϕ_λ which is of key importance in classifying the irreducible modules.

Lemma 1.1.3. *Let $\lambda \in \Lambda$. Then for any elements $S_1, S_2, T_1, T_2 \in M(\lambda)$, we have*

$$C^\lambda_{S_1,T_1} C^\lambda_{S_2,T_2} \equiv \phi_1(T_1, S_2) C^\lambda_{S_1,T_2} \mod A(< \lambda)$$

where $\phi_1 \in R$ is independent of T_2 and S_1.

Proof. This follows from [**8**, Lemma 1.7] in the case where $a = 1$ in that proof. $\square$

Definition 1.1.4 ([8, Definition 2.3]). *For $\lambda \in \Lambda$, define $\phi_\lambda : W(\lambda) \times W(\lambda) \to R$ by $\phi_\lambda(C_S, C_T) = \phi_1(S,T)$, where $S, T \in M(\lambda)$, and extend ϕ_λ bilinearly.*

From now on, we assume that R is a field. We now state one of the key results of the paper [**8**], which is the main one we aim to generalise to infinite dimensions. This result genuinely requires Λ to be finite.

Theorem 1.1.5 (Graham, Lehrer). *For $\lambda \in \Lambda$, the subspace* $\mathrm{rad}(\lambda)$ *of $W(\lambda)$ given by*

$$\mathrm{rad}(\lambda) := \{x \in W(\lambda) : \phi_\lambda(x,y) = 0 \text{ for all } \ y \in W(\lambda)\}$$

is an A-submodule of $W(\lambda)$. The quotient module $L(\lambda) := W(\lambda)/\mathrm{rad}(\lambda)$ is absolutely irreducible.

Define $\Lambda_0 := \{\lambda \in \Lambda : \phi_\lambda \neq 0\}$. Then the set $\{L(\lambda) : \lambda \in \Lambda_0\}$ is a complete set of equivalences classes of absolutely irreducible A-modules.

Proof. This follows from [**8**, Proposition 3.2, Theorem 3.4]. $\square$

1.2 Obstructions to cellular algebras in infinite dimensions.

Note that in the definition of a cellular algebra, there was no requirement that the set Λ be finite, although the sets $M(\lambda)$ for a fixed λ must be finite. (Removing the latter assumption would mean that the cell modules would be infinite dimensional, which would be very inconvenient.)

The following simple example (which is also discussed in the remarks following [**12**, Theorem 3.1]) shows that Theorem 1.1.5 fails fairly badly for a general cellular algebra with an infinite poset Λ.

Proposition 1.2.1. *Let R be an algebraically closed field and let A be the ring of polynomials $R[x]$. Let Λ be the set of natural numbers (including 0), ordered by the reverse of the usual order. Let $M(\lambda)$ be a set containing one element for each $\lambda \in \Lambda$. If $S \in M(\lambda)$, we define $C(S, S) = x^\lambda$. Let $*$ be the identity map on A.*

Then $(\Lambda, M, C, *)$ is a cell datum for A over R, in the sense of Definition 1.1.1.

Proof. We omit the proof, since it is almost trivial. □

Although this algebra $R[x]$ satisfies the cellular axioms, Theorem 1.1.5 fails severely when applied to it.

Proposition 1.2.2. *The irreducible representations of $R[x]$ over R are obtained by evaluation at x and are therefore in bijection with the elements of R.*

The set Λ_0 consists only of the element $0 \in \Lambda$.

Therefore Theorem 1.1.5 is false in general if Λ is infinite.

Proof. The first assertion follows from Schur's lemma, since R is algebraically closed.

The second assertion follows from the fact that $x^r.x^r \equiv 0 \mod \langle x^r \rangle$, unless $r = 0$ in which case $x^r.x^r \equiv x^r \mod \langle x^r \rangle$.

It is now obvious that the set Λ_0 does not index the absolutely irreducible $R[x]$-modules over R, which is the third assertion. □

We observe that the only irreducible representation of $R[x]$ which extends to a "continuous" representation of the power series ring $R[[x]]$ is the one which is indexed by the element $0 \in \Lambda_0$. This suggests that in order to find an analogue of Theorem 1.1.5 in the infinite dimensional case, one should consider modules for a suitable completion of the original algebra.

2. Procellular algebras

From now on, we assume that R is a field unless otherwise stated.

2.1 Construction of procellular algebras.

Definition 2.1.1. *A coideal I of the poset Λ is a subset of Λ for which $a \in I$ and $a < b$ imply $b \in I$.*

The coideal $\langle x_1, x_2, \ldots, x_r \rangle$ generated by the elements $x_1, x_2, \ldots, x_r$ consists of the elements of Λ given by

$$\{\lambda \in \Lambda : x_i \leq \lambda \text{ for some } 1 \leq i \leq r\}.$$

Definition 2.1.2. *We say that the cell datum $(\Lambda, M, C, *)$ for an algebra A is of profinite type if Λ is infinite and if for each $a \in \Lambda$, the coideal $\langle a \rangle$ is a finite set.*

The coideals of the poset Λ will be used to define cellular quotients of the infinite dimensional algebra A. We will then make an inverse system using these quotients. To achieve this, we make the following definitions.

Definition 2.1.3. *Let Λ be the poset of a cell datum of profinite type. We denote the set of finite coideals of Λ, ordered by inclusion, by Π.*

If $P \in \Pi$, we write A_P for the free R-module with basis parametrised by the set

$$\{C_P(S,T) : S, T \in M(\lambda), \lambda \in P\}.$$

We write I_P for the R-submodule of A spanned by all elements $C(S,T)$ where $S, T \in M(\lambda)$ for $\lambda \notin P$.

Proposition 2.1.4. *Let $P \in \Pi$ as above. Then A_P is a homomorphic image $\psi_P(A)$ of A corresponding to the quotient of A by the ideal I_P. The algebra A_P is a finite dimensional cellular algebra which inherits a cell datum from A by restriction of the poset Λ to the set P.*

Proof. This follows from the defining axioms for a cellular algebra. $\square$

Definition 2.1.5. *Maintain the above notation. Let $P_1, P_2 \in \Pi$ and suppose $P_1 \supset P_2$. We define the R-linear map $\psi_{P_1,P_2} : A_{P_1} \to A_{P_2}$ via the conditions*

$$\psi_{P_1,P_2}(C_{P_1}(S,T)) = \begin{cases} C_{P_2}(S,T) & \text{if } S,T \in M(\lambda) \text{ where } \lambda \in P_2, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.1.6. *The maps ψ of Definition 2.1.5 are surjective algebra homomorphisms.*

Proof. This follows from the fact that $\psi_{P_1,P_2}.\psi_{P_1} = \psi_{P_2}$. $\square$

We now use the maps ψ to define a inverse limit of cellular algebras. This idea has its roots in [**10**, §6.4].

Definition 2.1.7. *Let A be a cellular algebra with a cell datum of profinite type. Consider the inverse system whose elements are the sets A_P for $P \in \Pi$ and whose homomorphisms are given by*

$$\psi_{P_1,P_2} : A_{P_1} \to A_{P_2}$$

whenever $P_1 \supset P_2$. Denote the associated inverse limit by $\widehat{A}$. We call an algebra $\widehat{A}$ arising in this way a procellular algebra.

Remark. The terminology "procellular" is by analogy with profinite groups, which are inverse limits of finite groups.

2.2 Properties of procellular algebras.

An element of a procellular algebra $\widehat{A}$ can be regarded as a certain infinite combination of the basis elements $C(S,T)$ of A, as follows.

Proposition 2.2.1. *There is a canonical bijection between elements of $\widehat{A}$ and formal infinite sums*

$$\sum_{(T,T')} a(T,T')C(T,T'),$$

as (T,T') ranges over the domain of C.

The element of $\widehat{A}$ corresponding to this sum projects in the inverse system to the element of A_P given by

$$\sum_{(T,T')} a(T,T')\,\psi_P(C(T,T')).$$

Remark. The properties of the map ψ_P (see Proposition 2.1.4) mean that the second sum appearing above is finite.

Proof. By arguments familiar from §2.1, we see that the elements of A_P as defined above give (as P varies over Π) an element of $\widehat{A}$. It is clear that this correspondence is an injection.

It remains to show that every element of $\widehat{A}$ is of this form. Consider an element $C(T,T') \in A$, where $T, T' \in M(\lambda)$. Note that $\langle\lambda\rangle \in \Pi$. Define $a(T,T')$ to be the coefficient of $\psi_{\langle\lambda\rangle}(C(T,T'))$ in $A_{\langle\lambda\rangle}$. Then for any P such that $\lambda \in P$, the coefficient of $\psi_P(C(T,T'))$ in A_P must also be $a(T,T')$, by consideration of $\psi_{P,\langle\lambda\rangle}$. It follows that $\widehat{A}$ is determined solely by the coefficients $a(T,T')$, which proves the assertion. $\square$

Lemma 2.2.2. *The procellular algebra $\widehat{A}$ admits an algebra anti-automorphism $\widehat{*}$ of order 2 which arises from the anti-automorphisms $*_P$ on the cellular algebras A_P.*

Proof. This is immediate from the construction of the inverse system. $\square$

Definition 2.2.3. *Let $P \in \Pi$. We define $\widehat{I}_P$ to be the ideal of $\widehat{A}$ given by the infinite sums*

$$\sum a(T, T')C(T, T')$$

where $a(T, T') \neq 0 \Rightarrow T, T' \in M(\lambda)$ for $\lambda \notin P$.

The procellular algebra $\widehat{A}$ is equipped with a natural topology arising from the poset Λ. It will turn out that $\widehat{A}$ is in fact the completion of A with respect to this topology.

Proposition 2.2.4. *Let the set $\{\widehat{I}_P : P \in \Pi\}$ be a base of neighbourhoods of $0 \in \widehat{A}$. This gives $\widehat{A}$ the structure of a Hausdorff, complete topological ring in which the operation $\widehat{*}$ is a homeomorphism.*

Remark. By a topological ring, we mean a ring in which addition, negation and multiplication are continuous operations, so that in particular, addition is a homeomorphism. The topology may be specified by giving a base of neighbourhoods for 0, meaning that any open neighbourhood of 0 contains one of the neighbourhoods in the base. The open sets are those arising from the structure of the ring as a topological abelian group under addition.

Proof. The assertion that $\widehat{A}$ acquires the structure of a topological ring is clear, because the neighbourhoods in the base are ideals.

We show that the intersection of the ideals $\widehat{I}_P$ is trivial. Let $0 \neq x \in \bigcap_{P \in \Pi} \widehat{I}_P$, and suppose $C(T, T')$ appears with nonzero coefficient $a(T, T')$ in the infinite sum expansion of x. If $T, T' \in M(\lambda)$, then $\psi_{\langle\lambda\rangle}(x) \neq 0$, and therefore $x \notin \widehat{I}_{\langle\lambda\rangle}$, which is a contradiction.

It follows that $\widehat{I}_{\langle\lambda\rangle}$ and $x + \widehat{I}_{\langle\lambda\rangle}$ are disjoint open sets which contain the points 0 and x respectively. We deduce that the topology is Hausdorff. Completeness can be checked from the infinite sum realisation of $\widehat{A}$.

Finally we observe that $\widehat{*}$ is of order 2, is an algebra homomorphism, and fixes the ideals $\widehat{I}_P$ setwise. This proves the last assertion. $\square$

Corollary 2.2.5. *The algebra A embeds canonically in $\widehat{A}$ by considering the finite sums of $C(T, T')$ as special cases of the infinite sums of Proposition 2.2.1. Furthermore, $\widehat{A}$ is the completion of A with respect to the topology defined above.*

Proof. It is clear that the embedding works as stated. We also observe that any infinite sum can be approximated arbitrarily closely by a finite sum in the given topology, because all sets $P \in \Pi$ are finite. It follows that A embeds densely and thus that $\widehat{A}$ is the completion of A. $\square$

2.3 Representations of procellular algebras.

As might be expected from the theory of topological groups, we will concentrate on the "smooth" representations of the procellular algebras. We are also only interested in finite dimensional representations. We define the smooth representations as follows, based on [**1**, Definition 1.1].

Definition 2.3.1. *A representation ρ of $\widehat{A}$ on a space V is smooth if the annihilator of every vector of V is open.*

Lemma 2.3.2. *Let ρ be a finite dimensional representation of $\widehat{A}$ on a space V. Then ρ is smooth if and only if $\ker \rho$ is open.*

Proof. Suppose ρ is smooth. Then $\ker \rho$ is the intersection of a finite number of annihilators of vectors v and is therefore open.

Conversely, assume $\ker \rho$ is open, and pick $v \in V$. Then the annihilator of v is a union of cosets of $\ker \rho$, and is therefore open. $\square$

We will also refer to "smooth modules", with the obvious meaning.

Definition 2.3.3. *Let $\lambda \in \Lambda$. The cell module $\widehat{W}(\lambda)$ for $\widehat{A}$ is the left module obtained by considering the cell module $W(\lambda)$ for $A_{\langle \lambda \rangle}$ as a module for $\widehat{A}$ via the homomorphism $\psi_{\langle \lambda \rangle}$.*

The module $\widehat{W}(\lambda)$ inherits the bilinear form ϕ_λ from $W(\lambda)$. We define $\widehat{L}(\lambda)$ as the $\widehat{A}$-module corresponding to the $A_{\langle \lambda \rangle}$-module

$$L(\lambda) = W(\lambda)/\mathrm{rad}(\lambda).$$

The importance of the modules $\widehat{L}(\lambda)$ is demonstrated by the following theorem (compare with Theorem 1.1.5).

Theorem 2.3.4. *Let A be a cellular algebra with a datum of profinite type, and let $\widehat{A}$ be the corresponding procellular algebra.*

Define $\Lambda_0 := \{\lambda \in \Lambda : \phi_\lambda \neq 0\}$.

Then the set $\{\widehat{L}(\lambda) : \lambda \in \Lambda_0\}$ is a complete set of equivalences classes of absolutely irreducible smooth $\widehat{A}$-modules.

Proof. It is clear from the properties of $L(\lambda)$ that the module $\widehat{L}(\lambda)$ is absolutely irreducible for each λ.

To check the smoothness property, we observe that $\widehat{I}_{\langle \lambda \rangle}$ is an open set which lies in the kernel of the representation corresponding to $\widehat{W}(\lambda)$. The kernel of $\widehat{L}(\lambda)$ is a union of cosets of this open set and is therefore open. Thus by Lemma 2.3.2, the module is smooth.

We see that two modules $\widehat{L}(\lambda)$ and $\widehat{L}(\mu)$ are nonisomorphic by considering them as modules for the finite dimensional cellular algebra $A_{\langle \lambda, \mu \rangle}$.

It remains to prove that the given modules exhaust the set of absolutely irreducible smooth $\widehat{A}$-modules. Let L be such a module. Then the kernel of L is open, so $\widehat{I}_P$ annihilates L for some $P \in \Pi$. Thus L is an absolutely irreducible A_P-module in a canonical way. Proposition 2.1.4 shows that A_P is naturally a finite dimensional cellular algebra, which means by Theorem

1.1.5 that L is a cell module for A_P corresponding to some $\mu \in P$. The definition of the modules $\widehat{L}(\lambda)$ means that L, regarded as an $\widehat{A}$-module, is nothing other than $\widehat{L}(\mu)$. $\square$

3. Examples

It is now clear that the power series counterexample in Proposition 1.2.2 is compatible with Theorem 2.3.4. Any smooth module L for the procellular algebra $\widehat{R[x]} = R[[x]]$ must satisfy the property that $x^n.L = 0$ for sufficiently large n, meaning that $x.L = 0$. Thus the only smooth module for the procellular algebra corresponds to the element $0 \in \Lambda_0$.

This is a trivial example, but there are also interesting examples of procellular algebras as we explain in the next section.

3.1 The algebras $\dot{U}$.

In [**13**, §29], Lusztig essentially proves that the algebras $\dot{U}$, which are associated with Dynkin diagrams of various types, are cellular. (This result is referred to as the Peter–Weyl theorem.) The analogues of the finite dimensional algebras A_P are also considered. The algebras $\dot{U}$ are equipped with canonical bases, $\dot{B}$, which have many beautiful properties which are developed in [**13**].

We concentrate in this section on the case where $\dot{U}$ corresponds to a Dynkin diagram of type A, because this has interesting connections with the q-Schur algebra, and because it is better understood. Since we are only introducing these algebras for illustrative purposes, we shall not go into the full details. The interested reader is referred to the literature for more information.

The q-Schur algebra $S_q(n, r)$, which first appeared in [**2**], is a finite dimensional algebra over a base ring containing an indeterminate q and its square root $q^{1/2}$. This algebra was proved by the author to be cellular with respect to a Murphy-type basis [**10**, Proposition 6.2.1]. Du [**3**] introduced a canonical basis $\{\theta_{S,T}\}$ for this algebra, and the q-Schur algebra is also cellular with respect to the θ-basis.

Proposition 3.1.1. *The q-Schur algebra $S_q(n, r)$ has a basis $\{\theta_{S,T}\}$, where S, T are semistandard tableaux of the same shape with r boxes each and entries from the integers $\{1, 2, \ldots, n\}$.*

*This gives the q-Schur algebra the structure of a cellular algebra, where $\Lambda = \Lambda^+(n, r)$ ordered by the dominance order (as in [**9**, §2]), $M(\lambda)$ is the set of semistandard tableaux of shape λ (as above), C is the map $C(S, T) = \theta_{S,T}$ and $*$ is the anti-automorphism sending $\theta_{S,T}$ to $\theta_{T,S}$.*

Remark. Recall that "semistandard" means that the entries increase weakly along the rows, and strictly down the columns of the tableau.

This result is a generalisation of the example of the Hecke algebra of type A appearing in [**8**, Example 1.2].

Proof. Apart from the assertion involving $*$, this is simply [**6**, Theorem 5.3.3].

It follows from the definition of the θ-basis given in [**3**] that there is an anti-automorphism sending $\theta^{w}_{\lambda,\mu}$ to $\theta^{w^{-1}}_{\mu,\lambda}$, where λ, μ are compositions of r into at most n pieces, and w is a certain double coset representative in the symmetric group S_r.

The assertion about $*$ comes from properties of the Robinson–Schensted correspondence used in [**6**, §5.2]. It is a well-known property of this correspondence that inversion of the group element corresponds to exchange of the pair of tableaux. In the notation of semistandard tableaux, the anti-automorphism of the previous paragraph is nothing other than $*$. □

The terminology “canonical” for this basis is justified, as the following result shows.

Proposition 3.1.2 (Du). *For each r, there is an epimorphism of algebras*

$$\pi_r : \dot{U}(sl_n) \to S_q(n,r).$$

This takes elements of the basis $\dot{B}$ to zero or to canonical basis elements $\theta_{S,T}$.

For each basis element $b \in \dot{B}$, there exists an r such that $\pi_r(b) \neq 0$.

Proof. This follows from [**5**, Theorem 3.5] and the remarks in [**4**, §5.5]. □

Proposition 3.1.3. *Let Λ be the set of weights associated to finite dimensional simple modules for the Lie algebra sl_n over $\mathbb{C}$, ordered by the usual dominance order, where $\lambda < \mu$ means λ dominates μ.*

Let $M(\lambda)$ be the set of semistandard tableaux with l_i boxes in the i-th row, where $l_i \geq l_j$ for $i < j$, $l_n = 0$ and $\lambda_i = l_i - l_{i+1}$.

Let C be the map taking a pair (S,T) of semistandard tableaux of the same shape to the element $b \in \dot{B}$ corresponding to the canonical basis element $\theta_{S,T}$ of the q-Schur algebra $S_q(n,r)$ where $r = \sum \lambda_i$.

Let $*$ be the map induced by sending $\theta_{S,T}$ to $\theta_{T,S}$.

Then $(\Lambda, M, C, *)$ is a cell datum for $\dot{U}$, and the datum is of profinite type.

Note. One can check this directly using the theory of quantum groups developed in [**13**], but the proof we sketch below is based on inverse systems of q-Schur algebras.

Proof. The cell structure is essentially inherited from the cellular structure of the algebras $S_q(n, r)$ given in Proposition 3.1.1 and the homomorphisms of Proposition 3.1.2.

As mentioned in [**4**, §5.5], one can construct an inverse limit of q-Schur algebras using the epimorphisms ψ_{i+jn} from $S_q(n, i+(j+1)n)$ to $S_q(n, i+jn)$ which are dual to the q-analogue of coalgebra injection corresponding to multiplication by the determinant map. The effect of ψ on the θ-basis is known explicitly [**4**, 5.4 (b)] and takes a θ-basis element to another θ-basis element or to zero. The properties of cell ideals in cellular algebras now imply that if θ_{S_1,T_1} and θ_{S_2,T_2} are in the same cell (i.e. $S_1, T_1, S_2, T_2 \in M(\lambda)$ for the same λ), then their images under ψ are either both zero, or both nonzero and in the same cell.

It now follows that $\dot{U}$ does inherit a cellular structure from the inverse system described above. We can thus identify a basis element b with its images under the maps ψ, the "lowest" of which corresponds to a basis element θ whose tableaux have leftmost columns with fewer than n entries. The fact that the dominance order is the relevant order can be deduced from properties of the poset order in $S_q(n, r)$. Using these facts, we can check that the assertions in the statement hold. $\square$

Remark. One can check from the Robinson–Schensted correspondence that the effect of ψ is as follows. If S and T have a column of length n then S', T' are obtained from S, T respectively by removal of the leftmost columns, and $\theta_{S,T}$ maps to $\theta_{S',T'}$. If this is not the case then $\theta_{S,T}$ is mapped to zero.

This phenomenon of column removal was also studied in [**10**, §6.4].

3.2 Further remarks on procellular algebras.

The cell datum for $\dot{U}$ given in §3.1 gives rise to a procellular algebra $\widehat{U}$ which has interesting properties. Du [**4**, §5.5] calls the algebra $\widehat{U}$ the completion of U with respect to q-Schur algebras. The algebra $\widehat{U}$ has some convenient properties not shared by $\dot{U}$, such as the existence of a multiplicative identity.

The algebra $\dot{U}$ has a positivity property for its structure constants with respect to $\dot{B}$, which can be easily seen (by the results in §3.1) to be equivalent to the positivity of the structure constants for the θ-basis. The positivity property for the θ-basis was proved by the author in [**11**]. In both cases, the structure constants have interpretations in the geometrical framework of perverse sheaves.

The completion $\widehat{U}(sl_n)$, as well as containing the algebra $\dot{U}(sl_n)$, contains the quantized enveloping algebras $U(sl_n)$ in the sense of Drinfel'd and Jimbo. This was shown in [**10**, Theorem 6.4.19].

Several more of the main results concerning cellular algebras go over to procellular algebras essentially verbatim, so we will not discuss them explicitly here.

Another interesting example of a procellular algebra is the completion of the affine Temperley–Lieb algebra. In this case, the smooth modules for the completed algebra correspond to finite dimensional indecomposable modules of the original algebra. Furthermore, the procellular completion depends on a nonzero parameter; altering this parameter produces different families of indecomposable modules. The reader is referred to [**7**] for further details.

Acknowledgements

The author is grateful to Steffen König and Changchang Xi for conversations and correspondence relevant to this work, and to the referee for helpful suggestions.

References

[**1**] P. Cartier, *Representations of p-adic groups: a survey*, Proc. Sympo. Pure Math., **33** (1979), part 1, 111-155

[**2**] R. Dipper and G.D. James, *The q-Schur algebra*, Proc. L.M.S. **59** (1989), 23–50.

[**3**] J. Du, *Kazhdan-Lusztig bases and isomorphism theorems for q-Schur algebras*, Contemp. Math. **139** (1992), 121–140.

[**4**] J. Du, *q-Schur Algebras, Asymptotic Forms, and Quantum* SL_n, J. Alg **177** (1995), 385–408.

[**5**] J. Du, *Global IC Bases for Quantum Linear Groups*, J. Pure Appl. Alg. **114** (1996), 25–37.

[**6**] J. Du and H. Rui, *Based algebras and standard bases for quasi-hereditary algebras*, preprint.

[**7**] K. Erdmann and R.M. Green, *On Representations of Affine Temperley–Lieb Algebras, II*, Pac. J. Math., to appear

[**8**] J.J. Graham and G.I. Lehrer, *Cellular Algebras*, Invent. Math. **123** (1996), 1–34.

[**9**] J.A. Green, *Combinatorics and the Schur algebra*, J. Pure Appl. Alg. **88** (1993) 89–106.

[**10**] R.M. Green, Ph.D. thesis, University of Warwick, 1995.

[**11**] R.M. Green, *Positivity Properties for q-Schur Algebras*, Proc. Cam.

Phil. Soc., **122** (1997), 401–414.

[**12**] S. König and C. Xi, *On the structure of cellular algebras*, to appear in the Proceedings of the 8th International Conference on Representations of Algebras.

[**13**] G. Lusztig, *Introduction to Quantum Groups*, Birkhäuser, 1993.