A PROJECTION PROPERTY FOR KAZHDAN–LUSZTIG BASES

R.M. Green and J. Losonczy

Department of Mathematics and Statistics
Lancaster University
Lancaster LA1 4YF
England
E-mail: r.m.green@lancaster.ac.uk

Department of Mathematics
Long Island University
Brookville, NY 11548
USA
E-mail: losonczy@e-math.ams.org

Abstract. We compare the canonical basis for a generalized Temperley–Lieb algebra of type A or B with the Kazhdan–Lusztig basis for the corresponding Hecke algebra.

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Introduction

Generalized Temperley–Lieb algebras arise as certain quotients of Hecke algebras associated to Coxeter systems, in the same way that the ordinary Temperley–Lieb algebra can be realised as a quotient of the Hecke algebra of type A (see [**9**]). The finite dimensional generalized Temperley–Lieb algebras were classified by J. Graham [**5**] into seven infinite families: types A, B, D, E, F, H and I.

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In [7], we showed that a generalized Temperley–Lieb algebra arising from a Coxeter system of arbitrary type admits a canonical (more precisely, an IC) basis. Such a basis is by definition unique; furthermore, it is analogous, in a manner which can be made precise, to G. Lusztig's canonical basis for the $\pm$-part of a quantized enveloping algebra and also to the Kazhdan–Lusztig basis for a Hecke algebra. We determined the IC basis explicitly for the algebras of types A, D and E; in each case the basis coincides with a previously familiar basis with convenient properties [7, Theorem 3.6].

In this paper, we examine the relationship between the IC basis of a generalized Temperley–Lieb algebra and the corresponding Kazhdan–Lusztig basis, showing in particular that in type B, the projection of a certain natural subset of the Kazhdan–Lusztig basis agrees with the IC basis. Such a relationship is not obvious for a general Coxeter system. For example, the kernel of the relevant homomorphism need not be spanned by the Kazhdan–Lusztig basis elements which it contains. We point out that some of the results in this paper lead to a new, concise and more general proof of the main result of [4].

1. Preliminaries

1.1 Generalized Temperley–Lieb algebras and IC bases.

Let X be a Coxeter graph, of arbitrary type, and let $W(X)$ be the associated Coxeter group with distinguished set of generating involutions $S(X)$. Denote by $\mathcal{H}(X)$ the Hecke algebra associated to $W(X)$. (The reader is referred to [8, §7] for the basic theory of Hecke algebras arising from Coxeter systems.) Let $\mathcal{A}$ be the ring of Laurent polynomials, $\mathbb{Z}[v, v^{-1}]$. The $\mathcal{A}$-algebra $\mathcal{H}(X)$ has a basis consisting of elements T_w, with w ranging over $W(X)$, that satisfy

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ qT_{sw} + (q-1)T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

where ℓ is the length function on the Coxeter group $W(X)$, $w \in W(X)$, and $s \in S(X)$. The parameter q is equal to v^2.

Let $J(X)$ be the two-sided ideal of $\mathcal{H}(X)$ generated by the elements

$$\sum_{w \in \langle s, s' \rangle} T_w,$$

where (s, s') runs over all pairs of elements of $S(X)$ that correspond to adjacent nodes in the Coxeter graph. (If the nodes corresponding to (s, s') are connected by a bond of infinite strength, then we omit the corresponding relation.)

Definition 1.1.1. The generalized Temperley–Lieb algebra, $TL(X)$, is the quotient $\mathcal{A}$-algebra $\mathcal{H}(X)/J(X)$. We denote the canonical epimorphism of algebras by $\theta : \mathcal{H}(X) \longrightarrow TL(X)$.

Definition 1.1.2. A product $w_1 w_2 \cdots w_n$ of elements $w_i \in W(X)$ is called *reduced* if $\ell(w_1 w_2 \cdots w_n) = \sum_i \ell(w_i)$. We reserve the terminology *reduced expression* for reduced products $w_1 w_2 \cdots w_n$ in which every $w_i \in S(X)$.

Call an element $w \in W(X)$ *complex* if it can be written as a reduced product $x_1 w_{ss'} x_2$, where $x_1, x_2 \in W(X)$ and $w_{ss'}$ is the longest element of some rank 2 parabolic subgroup $\langle s, s' \rangle$ such that s and s' do not commute.

Denote by $W_c(X)$ the set of all elements of $W(X)$ that are not complex.

Let t_w denote the image of the basis element $T_w \in \mathcal{H}(X)$ in the quotient $TL(X)$.

Theorem 1.1.3 (Graham). *The set $\{t_w : w \in W_c\}$ is an $\mathcal{A}$-basis for the algebra $TL(X)$.*

Proof. See [**5**, Theorem 6.2]. □

The basis in Theorem 1.1.3 will be called the "t-basis". The t-basis plays a rôle in the definition of other important bases for the generalized Temperley–Lieb algebra.

Definition 1.1.4. For each $s \in S(X)$, we define $b_s = v^{-1} t_s + v^{-1} t_e$. If $w \in W_c(X)$ and $s_1 s_2 \cdots s_n$ is a reduced expression for w, then we define $b_w = b_{s_1} b_{s_2} \cdots b_{s_n}$.

Note that b_w does not depend on the choice of reduced expression for w.

It is known that $\{b_w : w \in W_c\}$ is a basis for the $\mathcal{A}$-module $TL(X)$. We shall call it the *monomial basis*.

We now recall a principal result of [**7**], which establishes the canonical basis for $TL(X)$. This basis is a direct analogue of the important Kazhdan–Lusztig basis of the Hecke algebra $\mathcal{H}(X)$.

Fix a Coxeter graph, X. Let $(I, \leq)$ be the poset with $I = W_c(X)$ and with $\leq$ defined as the restriction to I of the Bruhat–Chevalley order on $W(X)$.

Let $\mathcal{A}^- = \mathbb{Z}[v^{-1}]$. Let $^-$ be the involution on the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ which satisfies $\bar{v} = v^{-1}$.

By [**7**, Lemma 1.4], the algebra $TL(X)$ has a $\mathbb{Z}$-linear automorphism of order 2 which sends v to v^{-1} and t_w to $t_{w^{-1}}^{-1}$. We denote this map also by $^-$.

Let $\mathcal{L}$ be the free $\mathcal{A}^-$-submodule of $TL(X)$ with basis $\{v^{-\ell(w)} t_w : w \in W_c\}$, and let $\pi : \mathcal{L} \longrightarrow \mathcal{L}/v^{-1}\mathcal{L}$ be the canonical projection.

Theorem 1.1.5. *There exists a unique basis $\{c_w : w \in W_c\}$ for $\mathcal{L}$ such that $\overline{c_w} = c_w$ and $\pi(c_w) = \pi(v^{-\ell(w)} t_w)$ for all $w \in W_c$.*

Proof. This is [**7**, Theorem 2.3]. □

The basis $\{c_w : w \in W_c\}$ is called the *IC basis* (or the *canonical basis*) of $TL(X)$. It depends on the t-basis, the involution $^-$, and the lattice $\mathcal{L}$.

1.2 General questions.

It was shown in [**7**, Theorem 3.6] that the canonical basis of the generalized Temperley–Lieb algebra equals the monomial basis in types A, D and E. In this paper, we determine the IC basis for type B. Note that in any non-simply-laced case, the monomial basis does not equal the canonical basis [**7**, Remark 3.7 (1)]. It would be interesting to have a description of the canonical basis in each of the finite dimensional types.

A natural related problem concerns the relationship between the IC basis of a generalized Temperley–Lieb algebra and the Kazhdan–Lusztig basis for the corresponding Hecke algebra (see [**7**, Remark 2.4 (2)]). Recall from [**10**] that for each

$w \in W$ there exists a unique $C'_w \in \mathcal{H}$ such that $\overline{C'_w} = C'_w$, where $^-$ is a certain $\mathbb{Z}$-linear automorphism of $\mathcal{H}$, and

$$C'_w = \sum_{\substack{x \in W \\ x \leq w}} \widetilde{P}_{x,w}(v^{-\ell(x)} T_x),$$

where $\widetilde{P}_{x,w}$ lies in $v^{-1}\mathcal{A}^-$ if $x < w$, and $\widetilde{P}_{w,w} = 1$.

Let $\mathcal{C}$ denote the set of $C'_w \in \mathcal{H}$ indexed by $w \in W_c$. It is clear from [**7**, Lemma 1.5] that the set $\theta(\mathcal{C})$ is a basis for $TL(X)$.

Definition 1.2.1. We say that a Coxeter graph X satisfies the *projection property* if $\theta(\mathcal{C})$ equals the canonical basis of $TL(X)$.

We do not know of an example of a Coxeter graph which fails to have the projection property. The following two propositions provide useful sufficient conditions for $\theta(\mathcal{C})$ to equal the canonical basis.

Proposition 1.2.2. *The following are equivalent for a Coxeter graph X:*

(i) $\theta(C'_w) \in \mathcal{L}$ *for all* $w \in W(X)$*, where* $\mathcal{L}$ *is the lattice defined in §1.1.*

(ii) $\theta(v^{-\ell(w)} T_w) \in \mathcal{L}$ *for all* $w \in W(X)$.

If (i) or (ii) holds, then $\pi(\theta(C'_w)) = \pi(\theta(v^{-\ell(w)} T_w))$ *for all* $w \in W$*, so that* X *satisfies the projection property.*

Proof. For any $w \in W(X)$, we have

$$C'_w = \sum_{\substack{x \in W(X) \\ x \leq w}} \widetilde{P}_{x,w}(v^{-\ell(x)} T_x),$$

where $\widetilde{P}_{x,w}$ lies in $v^{-1}\mathcal{A}^-$ if $x < w$, and $\widetilde{P}_{w,w} = 1$. If we assume (ii), we see that $\theta(\widetilde{P}_{x,w}(v^{-\ell(x)} T_x)) \in v^{-1}\mathcal{L}$ if $x < w$, and $\theta(\widetilde{P}_{w,w}(v^{-\ell(w)} T_w)) = \theta(v^{-\ell(w)} T_w) \in \mathcal{L}$. Thus $\theta(C'_w) \in \mathcal{L}$ and $\pi(\theta(C'_w)) = \pi(\theta(v^{-\ell(w)} T_w))$, which proves that (ii) $\Rightarrow$ (i) and also that (ii) implies $\pi(\theta(C'_w)) = \pi(\theta(v^{-\ell(w)} T_w))$ for all $w \in W$.

Next, we show that (i) implies (ii). One sees from [**10**] that for any $w \in W(X)$ we have

$$v^{-\ell(w)} T_w = \varepsilon_w \sum_{\substack{x \in W \\ x \leq w}} \varepsilon_x \widetilde{Q}_{x,w} C'_x,$$

where $\varepsilon_x := (-1)^{\ell(x)}$, $\widetilde{Q}_{x,w}$ lies in $v^{-1}\mathcal{A}^-$ if $x < w$, and $\widetilde{Q}_{w,w} = 1$. An argument similar to that of the previous paragraph gives (i) $\Rightarrow$ (ii).

Finally, we verify that X satisfies the projection property if either (i) or (ii) holds. It was shown in [**7**, Lemma 1.4] that the automorphism $^-$ of $\mathcal{H}$ given in [**10**] induces the automorphism $^-$ of $TL(X)$ defined in §1.1. It follows that $\overline{\theta(C'_w)} = \theta(C'_w)$. Since $\pi(\theta(C'_w)) = \pi(\theta(v^{-\ell(w)}T_w))$, Theorem 1.1.5 shows that $\theta(C'_w) = c_w$ if $w \in W_c$, which completes the proof. $\square$

Proposition 1.2.3. *If X is a Coxeter graph such that the kernel of the canonical map $\theta : \mathcal{H}(X) \longrightarrow TL(X)$ is spanned by the basis elements C'_w which it contains, then X has the projection property.*

Proof. Suppose that the hypothesis is satisfied, and let $w \in W_c$. Recall from the proof of Proposition 1.2.2 that

$$v^{-\ell(w)}T_w = \varepsilon_w \sum_{\substack{x \in W \\ x \leq w}} \varepsilon_x \widetilde{Q}_{x,w} C'_x.$$

Applying θ to both sides yields

$$\theta(v^{-\ell(w)}T_w) = \varepsilon_w \sum_{\substack{x \in W_c \\ x \leq w}} \varepsilon_x \widetilde{Q}_{x,w} \theta(C'_x).$$

This means that the change of basis matrix from the basis $\{\theta(C'_w) : w \in W_c\}$ to the basis $\{v^{-\ell(w)}t_w : w \in W_c\}$ (with respect to some total refinement of the partial order $\leq$) is upper triangular with ones on the diagonal, and all the entries above the diagonal lie in $v^{-1}\mathcal{A}^-$. The inverse of this matrix has the same properties, meaning that

$$\theta(C'_w) = \sum_{\substack{x \in W_c \\ x \leq w}} \widetilde{P}_{x,w}(v^{-\ell(x)}t_x),$$

where $\widetilde{P}_{x,w} \in v^{-1}\mathcal{A}$ if $x < w$ and $\widetilde{P}_{w,w} = 1$. Thus $\pi(\theta(C'_w)) = \pi(v^{-\ell(w)}t_w) = \pi(c_w)$, and since $\theta(C'_w)$ is fixed by $^-$, the proposition follows. $\square$

We conclude this section with a conjecture (cf. [**7**, Remark 2.4 (1)]).

Conjecture 1.2.4. *Let X be an arbitrary Coxeter graph. Then the structure constants of the canonical basis for $TL(X)$ lie in $\mathbb{N}[v, v^{-1}]$.*

The conjecture follows easily in types A, D and E from the results in [**7**, §3].

It is not difficult to see that when $X = I_2(m)$ (the dihedral case), the hypothesis of Proposition 1.2.3 is satisfied: the ideal $J(I_2(m))$ is spanned by the single Kazhdan–Lusztig basis element C'_{w_0}, where w_0 denotes the longest element of $W(I_2(m))$. Moreover, it is known that the structure constants for the C'-basis of $\mathcal{H}(I_2(m))$ have positive coefficients. Thus, one is able to deduce the positivity property for the IC basis of the generalized Temperley–Lieb algebra $TL(I_2(m))$.

2. Types A and B

Throughout this section, we assume that X is a Coxeter graph of type A or B. The algebra $TL(X)$ is then generated by the monomial basis elements b_s, with s ranging over $S(X)$, subject to the relations $b_s^2 = q_c b_s$, $b_s b_{s'} = b_{s'} b_s$ if ss' has order 2, $b_s b_{s'} b_s = b_s$ if ss' has order 3, and $b_s b_{s'} b_s b_{s'} = 2 b_s b_{s'}$ if ss' has order 4 (see [**6**, §1]). Here, q_c is the Laurent polynomial $[2] = v + v^{-1}$.

Our goal is to prove that X has the projection property. This will be accomplished in Theorem 2.2.1.

2.1 Reduced expressions.

Our first lemma describes a normal form reduced expression for elements in a Coxeter group of type A or B.

Let $\sigma_1, \sigma_2, \dots, \sigma_r$ denote the Coxeter generators of the Coxeter group $W(B_r)$, where $\sigma_1\sigma_2$ has order 4 and $\sigma_i\sigma_{i+1}$ has order 3 for $i > 1$. Define $W^{(r)} = \{w \in W(B_r) : 1 \leq i < r \Rightarrow \ell(\sigma_i w) > \ell(w)\}$. Then $W^{(r)}$ is a set of minimum length right coset representatives for the subgroup $W(B_{r-1})$ of $W(B_r)$, and $\ell(xy) = \ell(x) + \ell(y)$ for all $x \in W(B_{r-1})$ and $y \in W^{(r)}$ (see [**8**, §5.12]). Furthermore, it is known that the elements of $W^{(r)}$ are given by

$$\{e,\ \sigma_r,\ \sigma_r\sigma_{r-1},\ \dots,\ \sigma_r\sigma_{r-1}\cdots\sigma_2\sigma_1,\ \sigma_r\sigma_{r-1}\cdots\sigma_2\sigma_1\sigma_2,\ \dots,$$

$$\sigma_r\sigma_{r-1}\cdots\sigma_2\sigma_1\sigma_2\cdots\sigma_{r-1}\sigma_r\}.$$

(This can be established by working with the signed permutation representation of $W(B_r)$.) Observe that each element of $W^{(r)}$ has a unique reduced expression.

Lemma 2.1.1. *Let $w \in W(X)$. There exists a reduced expression $s_1s_2\cdots s_n$ for w such that for each k, either s_k does not appear to the left of the k-th factor in $s_1s_2\cdots s_n$, or s_k does not commute with s_{k-1}.*

Proof. Every Coxeter system of type B_r contains a Coxeter system of type A_{r-1}, the Coxeter generators of the latter being a subset of those of the former. Hence, it is enough to treat the case $X = B_r$.

Considering the chain $\{e\} = W(B_0) \subseteq W(B_1) \subseteq \cdots \subseteq W(B_r) = W$ of Coxeter groups, we see that any $w \in W$ has a unique reduced decomposition $w = w_1w_2\cdots w_r$, where each $w_i \in W^{(i)}$. By substituting for each w_i its unique reduced expression, we obtain a normal form reduced expression for w which satisfies the condition in the statement of the lemma. □

Let $w \in W = W(X)$. It is known [**1**, §IV.1.5] that every reduced expression for w can be transformed into any other reduced expression for w by performing a sequence of braid moves. Given this fact, one may characterize W_c as the set of $w \in W$ such that every reduced expression for w can be transformed into any other reduced expression for w by a sequence of commutation moves (see [**11**, Proposition 1.1]).

We define the *content* of $w \in W$ to be the set $c(w)$ of Coxeter generators $s \in S$ that appear in some (any) reduced expression for w.

The next result was stated without proof in [**3**, §7.1]. The proof that we give is similar to that of [**2**, Lemma 2].

Lemma 2.1.2. *Let $w \in W_c$ and $s \in S$ satisfy $ws \notin W_c$. There exists a unique $s' \in S$ such that any reduced expression for w can be parsed in one of the following two ways.*

(i) $w = w_1 s w_2 s' w_3$, *where* ss' *has order* 3, *and* s *commutes with every member of* $c(w_2) \cup c(w_3)$*;*

(ii) $w = w_1 s' w_2 s w_3 s' w_4$, *where* ss' *has order* 4, s *commutes with every member of* $c(w_3) \cup c(w_4)$, *and* s' *commutes with every member of* $c(w_2) \cup c(w_3)$.

Proof. A sequence of commutation moves applied to any reduced expression $x = s_1 s_2 \cdots s_n$ results in a reduced expression $x = s_{\tau(1)} s_{\tau(2)} \cdots s_{\tau(n)}$, where the $\tau(i)$-th factor of $s_1 s_2 \cdots s_n$ has been moved to the i-th position.

We remark that if the generators $s_{\tau(i)}$ and $s_{\tau(j)}$ do not commute and $\tau(i) < \tau(j)$, then $i < j$.

Let $s_1 s_2 \cdots s_n$ be a reduced expression for w. We shall parse this reduced expression as in the statement of the lemma.

Define $s_{n+1} = s$. Since $s_1 s_2 \cdots s_{n+1} \notin W_c$, one can transform $s_1 s_2 \cdots s_{n+1}$ by a sequence of commutation moves into a reduced expression $s_{\tau(1)} s_{\tau(2)} \cdots s_{\tau(n+1)}$ which possesses a substring of type: (1) $s_{\tau(k)} s_{\tau(k+1)} s_{\tau(k+2)}$, where $s_{\tau(k)} = s_{\tau(k+2)}$ and $(s_{\tau(k)} s_{\tau(k+1)})^3 = 1$; or type (2) $s_{\tau(k-1)} s_{\tau(k)} s_{\tau(k+1)} s_{\tau(k+2)}$, where $s_{\tau(k-1)} = s_{\tau(k+1)}$, $s_{\tau(k)} = s_{\tau(k+2)}$ and $(s_{\tau(k)} s_{\tau(k+1)})^4 = 1$.

We claim that in either case, $\tau(k+2) = n+1$. By the previous remark, we have $n+1 \neq \tau(k), \tau(k+1)$, and also $n+1 \neq \tau(k-1)$ if we are considering a substring of type (2). Hence, if $\tau(k+2) \neq n+1$, then we may commute s_{n+1} to the end of $s_{\tau(1)} s_{\tau(2)} \cdots s_{\tau(n+1)}$ and thereby produce a reduced expression for ws which contains a substring of type (1) or (2) among the first n factors, contradicting $w \in W_c$. Thus, we have $s = s_{\tau(k)} = s_{\tau(k+2)}$.

Again by the remark, we have $\tau(k) < \tau(k+1) < \tau(k+2)$, and also $\tau(k-1) < \tau(k)$ if the substring is of type (2). Suppose that the substring is of type (1). Define $w_1 = s_1 s_2 \cdots s_{\tau(k)-1}$, $w_2 = s_{\tau(k)+1} s_{\tau(k)+2} \cdots s_{\tau(k+1)-1}$, $s' = s_{\tau(k+1)}$ and $w_3 = s_{\tau(k+1)+1} s_{\tau(k+1)+2} \cdots s_n$. Observe that in order for the substring of type (1) to have occurred, it must be true that s commutes with every member of $c(w_2) \cup c(w_3)$. (To see why, consider the sequence of commutation moves required to transform the subexpression $s w_2 s' w_3 s$ of ws into an expression where the occurrences of s,

s', s are consecutive.)

If instead the substring is of type (2), then we define $w_1 = s_1 s_2 \cdots s_{\tau(k-1)-1}$, $s' = s_{\tau(k-1)}$, $w_2 = s_{\tau(k-1)+1} s_{\tau(k-1)+2} \cdots s_{\tau(k)-1}$, $w_3 = s_{\tau(k)+1} s_{\tau(k)+2} \cdots s_{\tau(k+1)-1}$ and $w_4 = s_{\tau(k+1)+1} s_{\tau(k+1)+2} \cdots s_n$. As above, in order for the substring of type (2) to have occurred, it must be the case that s commutes with every member of $c(w_3) \cup c(w_4)$ and s' commutes with every member of $c(w_2) \cup c(w_3)$.

Regardless of the type of substring that arises, inspection of the parsed form of the original reduced expression reveals that s' is the rightmost factor which does not commute with s in any reduced expression for w. Thus, s' is uniquely determined by w and s. $\square$

Before presenting the next lemma, which furnishes some relevant information concerning the structure constants of the monomial basis, we remark that if $w = s_1 s_2 \cdots s_n$ is a reduced expression and $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, then the product $b_{s_{i_1}} b_{s_{i_2}} \cdots b_{s_{i_k}}$ equals $a q_c^m b_x$, where a and m are nonnegative integers, and $x \in W_c$ satisfies $\ell(x) \leq k$ and $x \leq w$. This follows easily from the subexpression characterization of Bruhat–Chevalley order and the presentation of $TL(X)$ given at the beginning of this section. We shall invoke this fact freely in the remainder of §2.

We remind the reader that the elements $b_w \in TL(X)$ are always understood to be indexed by group elements $w \in W_c$ (see Definition 1.1.4).

Lemma 2.1.3. *Let $w \in W_c$ and $s \in S$. Then $b_w b_s = a q_c^m b_{w'}$ for some $w' \in W_c$ and some nonnegative integers a and m, where $m \leq 1$. We have $\ell(w's) < \ell(w')$. If $\ell(ws') < \ell(w)$ for some $s' \in S$ which does not commute with s, then $m = 0$.*

Proof. We first address the last assertion in the statement. Suppose that w has a reduced expression ending in a generator s' which does not commute with s. Since $w \in W_c$, this implies $\ell(ws) > \ell(w)$. If $ws \notin W_c$ then by Lemma 2.1.2, it is possible to write either $w = uss'$ (reduced), if $(ss')^3 = 1$, or $w = us'ss'$ (reduced), if $(ss')^4 = 1$. In either case, $b_w b_s$ equals a positive integer multiple of some monomial basis element.

We now prove the rest of the lemma by induction on $\ell(w)$. The statement is true for $\ell(w) \leq 1$, so we may assume $\ell(w) > 1$.

Suppose that $\ell(ws) < \ell(w)$. We may write $w = us$ (reduced) for some $u \in W_c$. We then have $b_w b_s = b_u b_s b_s = q_c b_u b_s = q_c b_w$, and the induction step follows.

Suppose instead that $\ell(ws) > \ell(w)$. If $ws \in W_c$, the result is obvious: set $m = 0$, $a = 1$ and $w' = ws$. If $ws \notin W_c$ then we choose a reduced expression for w as in the statement of Lemma 2.1.1, and parse it according to Lemma 2.1.2. We treat only the case where the reduced expression may be parsed as $w = w_1 s' w_2 s w_3 s' w_4$, where $(ss')^4 = 1$, s commutes with every member of $c(w_3) \cup c(w_4)$, and s' commutes with every member of $c(w_2) \cup c(w_3)$. The reasoning for the other case is similar.

If $\ell(w_4) = 0$, then $b_w b_s = b_{w_1 w_2 w_3} b_{s'} b_s b_{s'} b_s = 2\, b_{w_1 w_2 w_3 s' s}$, and the induction step follows.

If $\ell(w_4) = 1$, then we have $b_w b_s = b_{w_1 w_2 w_3} b_{s'} b_s b_{s'} b_{w_4} b_s = b_{w_1 w_2 w_3} b_{s'} b_s b_{s'} b_s b_{w_4} = 2\, b_{w_1 w_2 w_3 s' s} b_{w_4}$. By induction, the last expression equals $a q_c^m b_{w'}$, where $m \leq 1$. Also, since $b_w b_s = 2 b_{w_1 w_2 w_3 s'} b_{w_4} b_s$, induction gives $\ell(w's) < \ell(w')$.

If $\ell(w_4) > 1$, then we parse $w_4 = w_5 s_1 s_2$ ($s_1, s_2 \in S$) according to the chosen reduced expression for w, and then we apply the inductive hypothesis to $b_{ws_2} b_s$ (note that $\ell(ws_2) < \ell(w)$ and so $b_{ws_2} b_{s_2} = b_w$), finding that it equals $a q_c^m b_{w'}$, where $m \leq 1$ and $\ell(w's) < \ell(w')$. Since our chosen reduced expression for w conforms to the condition in Lemma 2.1.1, either $s_2 \notin c(ws_2)$ or s_2 does not commute with s_1. In the first case, we have

$$b_w b_s = b_{ws_2} b_{s_2} b_s = b_{ws_2} b_s b_{s_2} = a q_c^m b_{w'} b_{s_2} = a q_c^m b_{w' s_2},$$

where the last equality holds because $s_2 \notin c(w')$. Since s commutes with s_2, we have $\ell(w' s_2 s) < \ell(w' s_2)$.

For the case where s_1 and s_2 do not commute, we observe that

$$a q_c^m b_{w'} = b_{ws_2} b_s = b_{w_1 w_2 w_3} b_{s'} b_s b_{s'} b_{w_5} b_{s_1} b_s = 2\, b_{w_1 w_2 w_3 s' s} b_{w_5} b_{s_1}.$$

By induction, w' has a reduced expression ending in s_1. Hence, by the argument given in the first paragraph, if $w's_2 \notin W_c$, then $b_{w'}b_{s_2}$ equals $a'b_{w''}$ for some positive integer a' and some $w'' \in W_c$, so that $b_w b_s = b_{ws_2}b_{s_2}b_s = b_{ws_2}b_s b_{s_2} = aq_c^m b_{w'}b_{s_2} = aa'q_c^m b_{w''}$. On the other hand, since

$$b_w b_s = b_{w_1w_2w_3}b_{s'}b_s b_{s'}b_{w_4}b_s = b_{w_1w_2w_3}b_{s'}b_s b_{s'}b_s b_{w_4} = 2\, b_{w_1w_2w_3s'}b_{w_4}b_s,$$

induction gives $\ell(w''s) < \ell(w'')$.

The induction step is complete. □

If a reduced expression satisfies the condition in the statement of the following lemma, then we say that it has the *deletion property.*

Lemma 2.1.4. *Let $w \in W$. There exists a reduced expression $s_1s_2\cdots s_n$ for w such that for any $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, we have $b_{s_{i_1}}b_{s_{i_2}}\cdots b_{s_{i_k}} = aq_c^m b_x$ for some $x \in W_c$ and some nonnegative integers a and m, where $m \leq n-k$.*

Proof. We shall prove by induction on $n = \ell(w)$ that if a reduced expression $s_1s_2\cdots s_n$ for w satisfies the condition in the statement of Lemma 2.1.1, then it has the deletion property. When $n \leq 2$, this is evidently true, so we may assume $n > 2$. Given $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, we consider two cases.

Case 1: $i_{k-1} < n-1$. We apply the inductive hypothesis to $s_1s_2\cdots s_{n-2}$ and the integers $1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq n-2$, thereby obtaining $b_{s_{i_1}}b_{s_{i_2}}\cdots b_{s_{i_{k-1}}} = aq_c^m b_x$, where $m \leq (n-2)-(k-1)$. By Lemma 2.1.3, we have $b_{s_{i_1}}b_{s_{i_2}}\cdots b_{s_{i_k}} = aq_c^m b_x b_{s_{i_k}} = a'q_c^{m'}b_{x'}$, where $m' \leq m+1 \leq n-k$.

Case 2: $i_{k-1} = n-1$. Here, we apply induction to $s_1s_2\cdots s_{n-1}$ and the integers $1 \leq i_1 < i_2 < \cdots < i_{k-1} = n-1$. We obtain $b_{s_{i_1}}b_{s_{i_2}}\cdots b_{s_{i_{k-1}}} = aq_c^m b_x$, where $m \leq (n-1)-(k-1)$. If $s_n \notin c(x)$, then the conclusion follows after multiplying both sides of the last equation on the right by b_{s_n}. If instead $s_n \in c(x)$, then because our reduced expression for w satisfies the condition in Lemma 2.1.1, s_{n-1} and s_n do not commute. Furthermore, by Lemma 2.1.3, the element $x \in W_c$ has a reduced expression ending in $s_{i_{k-1}} = s_{n-1}$, hence by the last assertion of the same lemma, $b_x b_{s_n}$ equals a nonnegative integer multiple of some monomial basis element. □

We note that Lemma 2.1.4 may also be proved using the diagram calculus for $TL(B_n)$ described in [**6**, Theorem 4.1] in conjunction with Lemma 2.1.1.

2.2 Canonical basis in types A and B.

We are now prepared to state and prove the main result of this paper, which can be viewed as a generalization of [**4**, Theorem 3.8.2].

Theorem 2.2.1. *Let X be a Coxeter graph of type A or B. Then the canonical basis of $TL(X)$ equals the image of the set of Kazhdan–Lusztig basis elements $C'_w \in \mathcal{H}(X)$ indexed by $w \in W_c(X)$.*

Proof. By Proposition 1.2.2 (ii), it suffices to show that for all $w \in W$, the element $v^{-\ell(w)} t_w$ lies in the lattice $\mathcal{L}$.

Every $w \in W$ has a reduced expression $s_1 s_2 \cdots s_n$ as in the statement of Lemma 2.1.4. We have

$$\begin{aligned} v^{-\ell(w)} t_w &= (v^{-1} t_{s_1})(v^{-1} t_{s_2}) \cdots (v^{-1} t_{s_n}) \\ &= (b_{s_1} - v^{-1})(b_{s_2} - v^{-1}) \cdots (b_{s_n} - v^{-1}). \end{aligned}$$

The last product expands to a sum of terms

$$\pm v^{k-\ell(w)} b_{s_{i_1}} b_{s_{i_2}} \cdots b_{s_{i_k}},$$

and each such term equals

$$a v^{k-\ell(w)} q_c^m b_x,$$

where $a \in \mathbb{Z}$, $x \in W_c$ satisfies $x \leq w$, and $m \leq \ell(w) - k$ by Lemma 2.1.4. The coefficient of b_x in the expansion therefore lies in $\mathcal{A}^-$. Also, one sees that if $w \in W_c$, then the coefficient of b_w equals 1.

We have shown that for any $w \in W$, the element $v^{-\ell(w)} t_w$ equals a linear combination of b_x ($x \in W_c$ and $x \leq w$) with coefficients in $\mathcal{A}^-$; and if $w \in W_c$, then the coefficient of b_w equals 1. In particular, the change of basis matrix from the monomial basis $\{b_x : x \in W_c\}$ to the basis $\{v^{-\ell(x)} t_x : x \in W_c\}$ (with respect to some total refinement of $\leq$) is upper triangular with entries in $\mathcal{A}^-$ and with ones on the diagonal. The inverse of this matrix must also have this property. That is, for any $x \in W_c$, the element b_x equals a linear combination of $v^{-\ell(y)} t_y$ ($y \in W_c$ and $y \leq x$) with coefficients in $\mathcal{A}^-$. Combining this with the first statement of the present paragraph, we conclude that every $v^{-\ell(w)} t_w$ lies in $\mathcal{L}$. $\square$

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