ON THE AFFINE TEMPERLEY–LIEB ALGEBRAS

C.K. Fan and R.M. Green

Institute for Advanced Study
Princeton, NJ 08540
USA
E-mail: ckfan@math.harvard.edu

Mathematical Institute
Oxford University
24–29 St. Giles'
Oxford OX1 3LB
England
E-mail: greenr@maths.ox.ac.uk

Abstract. We describe the cell structure of the affine Temperley–Lieb algebra with respect to a monomial basis. We construct a diagram calculus for this algebra.

1. Introduction

Let Γ be a Coxeter graph of type $\widehat{A}_l$, with $l > 1$. Let W be the associated Coxeter group with generating set S. Recall that S consists of $n = l+1$ generators, one for each node in Γ. These generators satisfy the relations $s^2 = 1$, $st = ts$ if s and t are not connected in Γ, and $sts = tst$ if s and t are connected in Γ.

Every $w \in W$ may be written as a product $s_1 s_2 s_3 \cdots s_m$ of generators in S. If m is minimal, we call this product "reduced" and define $\ell(w) = m$. More generally, if $w = w_1 w_2 w_3 \cdots w_m$ satisfies $\ell(w) = \sum_i \ell(w_i)$, then we call this product "reduced" as well.

Let H be the Iwahori-Hecke algebra associated to W. This is an algebra over $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ with generators T_s for each $s \in S$ and relations $T_s^2 = (v^2-1)T_s + v^2$, $T_s T_t = T_t T_s$ if $st = ts$, and $T_s T_t T_s = T_t T_s T_t$ if $sts = tst$, where s, $t \in S$. This

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algebra has a basis T_w, $w \in W$, where we have $T_w = T_{s_1} \cdots T_{s_m}$ whenever $s_1 \cdots s_m$ is a reduced expression for w.

Let I be the two-sided ideal generated by the elements

$$T_{sts} + T_{st} + T_{ts} + T_s + T_t + 1$$

where we have one such expression for each pair of non-commuting generators s, $t \in S$.

Define $TL(\widehat{A}_l) = H/I$.

The algebra $TL(\widehat{A}_l)$ is infinite dimensional and is the affine version of the Temperley–Lieb algebra. It has already appeared, in the case where l is odd, in the physics literature ([**7**], [**8**]). The purpose of this paper is to understand the structure of this algebra.

We shall do this in two ways. In the first part of the paper, we shall construct cells with respect to a basis of monomials and analyze the cell structure. This method of attack has its roots in [**5**] and [**6**], but it should be noted that the approach we adopt here is much more elementary and does not involve Kazhdan–Lusztig theory.

In the second part of this paper, we shall give an interpretation of $TL(\widehat{A}_l)$ as an algebra of certain diagrams about a cylinder. These diagrams turn out to give the same basis as the monomial basis mentioned above. This is a generalization of the diagram calculus for the ordinary Temperley–Lieb algebra, described for example in [**10**]. It is also a generalization of the diagram calculus for Jones's annular algebra (see [**4**]), which is a certain finite dimensional algebra related to the Hecke algebra of type $\widehat{A}$. (The relationship between the affine Temperley–Lieb algebra and the Jones algebra is explored in depth in [**3**].) The idea of representing the affine Temperley–Lieb algebra by diagrams inscribed on a cylinder has been used in [**7**], but that paper does not prove that the representation is faithful. This is the main result of the second part of the paper, Theorem 4.5.2.

2. Preliminaries

2.1 Some notation.

Let $w \in W$. Define the support of w to be the set of generators which appear in some (or any) reduced word for w. We shall also denote this by $\mathrm{supp}(w)$. Define

$\mathcal{L}(w)$ to be the set of $s \in S$ such that $\ell(sw) < \ell(w)$. Similarly, define $\mathcal{R}(w)$ to be the set of $s \in S$ such that $\ell(ws) < \ell(w)$.

Denote by W_c those elements of W whose reduced expressions avoid substrings of the form sts where s and t are non-commuting generators in S.

Elements of W_c enjoy the following property, proven in [**1**].

Property R. *Let $w \in W_c$, $t \in S$. Assume $wt \notin W_c$. Then we may write $w = w_1tsw_2$ reduced, t commutes with all $u \in \mathrm{supp}(w_2)$, and $(st)^3 = 1$. Furthermore, if $w = w_1'ts'w_2'$ is another such expression, then $s = s'$.*

We shall refer to the dual statement of property R, which involves generators being written in opposite order, also as property R.

For each simple generator $s \in S$, let E_s be the projection of $v^{-1}(T_s + 1) \in H$ to $TL(\widehat{A}_l)$. For any $w \in W_c$, it makes sense to define $E_w = E_{s_1} \cdots E_{s_n}$, where $s_1 \cdots s_n$ is any reduced expression for w. It was shown in [**1**] that the E_w, $w \in W_c$, constitute a basis for $TL(\widehat{A}_l)$, and that $TL(\widehat{A}_l)$ is generated by the E_s, $s \in S$, with the following relations: $E_s^2 = [2]E_s$, $E_sE_t = E_tE_s$ if $st = ts$, and $E_sE_tE_s = E_s$ if $sts = tst$, $s \neq t$, where $[2] = v + v^{-1}$.

Definition 2.1.1. Let $w \in W_c$. Let $s \in \mathcal{L}(w)$. Suppose that there exists $t \in S$ such that $sts = tst$ and $E_tE_w = E_{sw}$. We shall then say that s can be left cancelled, or that s is left cancellable, and that s is left cancellable by t.

We make an analogous definition for $s \in \mathcal{R}(w)$.

Example 2.1.2. Consider $TL(\widehat{A}_4)$. Label the simple generators $s_1, \ldots, s_5$, where s_i and s_j commute if and only if $1 < |i-j| < 4$. In the element $w = s_1s_3s_2s_4$, the generator s_3 is in $\mathcal{L}(w)$ and is left cancellable, for $E_{s_4}E_w = E_{s_1s_2s_4}$. Similarly, s_2 is right cancellable by s_1.

Let P denote the set of subsets of Γ which consist of non-adjacent nodes. We allow P to include the empty set, $\emptyset$. For any $U \in P$, let $\iota(U)$ be the product of the simple generators corresponding to the nodes in U (with $\iota(\emptyset) = 1$). Note that the order of the product is immaterial since the nodes in U are non-adjacent.

2.2 Cells.

We now imitate the definitions of left, right, and two-sided cells given by Kazhdan and Lusztig in [**5**]. The following definitions have already been used in [**2**]. Because of the simplicity of the structure constants for the basis $\{E_w\}$, the definitions of the various cells is simpler than that in [**5**]. We remark that these cells do not agree in general with the Kazhdan-Lusztig cells for the affine Weyl group W which are described in [**9**].

For any w, $w' \in W_c$, we say that $w' \leq_L w$ if there exists E_x such that $E_x E_w = cE_{w'}$ where $c \neq 0$.

For any w, $w' \in W_c$, we say that $w' \leq_R w$ if there exists E_x such that $E_w E_x = cE_{w'}$ where $c \neq 0$.

For any w, $w' \in W_c$, we say that $w' \leq_{LR} w$ if there exist E_x, E_y such that $E_x E_w E_y = cE_{w'}$ where $c \neq 0$.

We say $w \sim_L w'$ if both $w' \leq_L w$ and $w \leq_L w'$. Similarly, we define $w \sim_R w'$ and $w \sim_{LR} w'$.

The equivalence classes of $\sim_L$, $\sim_R$, and $\sim_{LR}$ are respectively, the left, right, and two-sided cells of W_c. Define $\mathbf{L}(w) = \{w' \in W_c \mid w' \sim_L w\}$, $\mathbf{R}(w) = \{w' \in W_c \mid w' \sim_R w\}$, and $\mathbf{T}(w) = \{w' \in W_c \mid w' \sim_{LR} w\}$.

2.3 The a-function.

The following function plays the analogous role in our situation as the Lusztig a-function (see [**6**]) does for Kazhdan-Lusztig cells.

For any $w \in W_c$, we define a nonnegative integer $a(w)$ as follows.

Definition 2.3.1. Let $w \in W_c$. Define

$$a(w) = \max_{U \in P} \{\#U \mid \iota(U) \text{ is a substring of some reduced expression for } w\}.$$

Observe that if $\ell(sw) > \ell(w)$, then $a(sw) \geq a(w)$ since any substring of a reduced expression for w would be a substring for a reduced expression for sw. Also note that $a(\iota(U)) = \#U$.

2.4 Left and right decompositions.

Let $w \in W_c$.

Note that we can uniquely write $w = \iota(G_1)\cdots\iota(G_m)$ reduced, where $G_k = \mathcal{L}(\iota(G_{k-1})\cdots\iota(G_1)w) \in P$ and $G_m \neq \emptyset$. Associated to this decomposition, we can define graphs γ_k, $1 \leq k < m$, where γ_k is the subgraph of Γ which includes the nodes in G_k and G_{k+1} along with all edges whose endpoints are in $G_k \cup G_{k+1}$. We shall refer to the sequence (G_k, γ_k) as the left decomposition of w.

Similarly, we can define the right decomposition. All the statements we make about the left decomposition have a natural counterpart for the right decomposition.

Lemma 2.4.1. *Let $w \in W_c$ be such that $w = x\iota(G_1)\iota(G_2)\iota(G_3)x'$ reduced. We have $s \in G_1 \cap G_3$ if and only if s fails to commute with 2 generators in G_2. That is, G_2 contains both generators which do not commute with s.*

Proof. Observe that $s\iota(G_2)s$ occurs as a reduced subword of some reduced expression for w. If the lemma is false, then either w is not reduced or it is not in W_c. $\square$

3. The structure of cells

3.1 The set Q.

Let $Q \subset W_c$ denote those elements w of W_c such that no element of $\mathcal{L}(w)$ is left cancellable and no element of $\mathcal{R}(w)$ is right cancellable. This leads to the following lemma.

Lemma 3.1.1. *Every two-sided cell contains an element of Q.*

Proof. Let T be any two-sided cell, and pick any $w \in T$. If $w \notin Q$, then we can cancel a simple generator $s \in S$ either from the left or from the right of w. We continue left or right cancelling until we arrive at an element of Q. The statement follows. $\square$

When n is even, there are two elements of P (see 2.1) of maximal cardinality. Let M_1 and M_2 denote these maximal elements of P. In this case, denote by M the set of elements of the form $(\iota(M_a)\iota(M_b))^k\iota(M_a)$ and $(\iota(M_a)\iota(M_b))^{k+1}$, for $k \geq 0$ and positive integers a, b such that $ab = 2$. Note that M is nonempty only when n is even.

Proposition 3.1.2. *The set Q consists of the elements $\iota(T)$, $T \in P$ and, additionally, in the case where n is even, the elements of M.*

Proof. First note that all the elements listed in the statement of the proposition belong to Q.

Let $w \in Q$. Let (G_k, γ_k), $1 \leq k \leq m$, be the left decomposition of w.

If $m = 1$, there is nothing to prove, so assume $m > 1$.

Because no element of $\mathcal{L}(w)$ can be left cancelled, all the nodes in G_2 must be connected to two nodes of G_1. There are two cases. Either γ_1 is a union of graphs of type A, or γ_1 is a graph of type $\widehat{A}$.

Suppose γ_1 is a union of graphs of type A. We must have that each connected component of γ_1 has one more node in G_1 than in G_2. By lemma 2.4.1, we may conclude that γ_2 is also a union of graphs of type A, each component of which has one more node in G_2 than in G_3. Continued application of lemma 2.4.1 shows that γ_{m-1} is a union of graphs of type A, each component of which has one more node in G_{m-1} than in G_m. However, this implies that an element of $G_m \subset \mathcal{R}(w)$ is right cancellable contrary to the assumption that $w \in Q$.

Suppose γ_1 is of type $\widehat{A}$. By the argument used to eliminate the case where γ_1 is a union of graphs of type A, we can conclude that $\gamma_k = \Gamma$ for all $1 \leq k < m$. It follows that each G_k is either M_1 or M_2, and, since the left decomposition is reduced, we have $G_k \neq G_{k+1}$.

The proposition follows. $\square$

We now define an equivalence relation $\sim$ on Q. We begin with a lemma.

Lemma 3.1.3. *Let q, $q' \in Q$. Suppose there exists $s \in S$ such that $E_q = E_s E_{q'} E_s$. Then there exists $t \in S$ such that $E_{q'} = E_t E_q E_t$.*

Proof. Note that if n is even and $q' \in M$, then there is no $s \in S$ such that $E_q = E_s E_{q'} E_s$.

Therefore $q' = \iota(T)$ for some $T \in P$ such that $\#T < n/2$. Since $E_q = E_s E_{q'} E_s$, we cannot have $s \in T$ and we cannot have that s commutes with all generators in T. If s fails to commute with two generators in T, then $q = sq's$ reduced, which contradicts proposition 3.1.2. Therefore, s fails to commute with exactly one generator $t \in T$ and $q = \iota(T \cup \{s\} \setminus \{t\})$.

We compute that $E_{q'} = E_t E_q E_t$. The lemma follows. $\square$

We say that q and $q' \in Q$ are neighbours if and only if there exists $s \in S$ such that $E_q = E_s E_{q'} E_s$. The previous lemma shows that this is symmetric. Define $\sim$ to be the transitive closure of the relation of being neighbours.

Proposition 3.1.4. *The equivalence classes of $\sim$ are as follows.*

1. $\{\iota(T) \mid T \in P, \#T = k\}$ *where k is a fixed integer less than $n/2$.*
2. *The one element sets $\{q\}$ for $q \in M$ (if n is even).*

We omit the proof.

3.2 Multiplication.

We prove a number of facts about the algebra multiplication.

Lemma 3.2.1. *Let $q \in Q$. Let $s_1, s_2, s_3, \ldots, s_a$ and $t_1, t_2, t_3, \ldots, t_b$ be elements of S. Define w by $E_{s_a} \cdots E_{s_1} E_q E_{t_1} \cdots E_{t_b} = cE_w$, $c \in \mathcal{A}$. Then there exists $q' \in Q$ such that $q' \sim q$ and some reduced expression for w involves a substring equal to q'.*

Proof. Let $w' = xqy$ be reduced and $s \in S$. Define w'' by $cE_{w''} = E_s E_{w'}$.

If $sw' \in W_c$ then either $w'' = w'$ or $w'' = sw'$ depending on whether $\ell(sw') < \ell(w')$ or $\ell(sw') > \ell(w')$, respectively. In either case, w'' has a reduced expression with a substring equal to q.

Assume $sw' \notin W_c$. Using property R, we argue as follows.

If s commutes with all generators in $\mathrm{supp}(xq)$, then $E_s E_{w'} = E_x E_q E_s E_y$. Note that $E_s E_y$ may be written as a product of $\ell(y) - 1$ basis elements E_{s_k} for $s_k \in S$.

If s commutes with all generators in $\mathrm{supp}(x)$, but not with all the generators in $\mathrm{supp}(q)$, then we must have $y = sy'$ and s commutes with all but one $t \in \mathrm{supp}(q)$. Let $T = (\mathrm{supp}(q) \cup \{s\}) \setminus \{t\}$. Observe that $\iota(T) \sim q$. We compute that $E_s E_{w'} = E_x E_{\iota(T)} E_{y'}$.

If s does not commute with all the generators in $\mathrm{supp}(x)$, then $E_s E_x$ can be written as a product of $\ell(x) - 1$ basis elements E_{s_k} for $s_k \in S$.

By symmetry and induction on the length of the sequence s_k, these observations imply the lemma. $\square$

We have the following corollary of lemma 3.2.1.

Corollary 3.2.2. *Let $w \in W_c$, $s \in S$. Define w' by $cE_{w'} = E_w E_s$, $c \in \mathcal{A}$. Then $a(w') \geq a(w)$.*

Lemma 3.2.1 can be strengthened in the following situation.

Lemma 3.2.3. *Let $q \in Q$. Let $s_1, s_2, s_3, \ldots, s_m$ be elements of S. Define $w' \in W_c$ by $E_q E_{s_1} E_{s_2} E_{s_3} \cdots E_{s_m} = cE_{w'}$, $c \in \mathcal{A}$. Then one can write $w' = qx$ reduced.*

Proof. We proceed by induction on m, the case where $m = 0$ being clear.

Suppose the lemma is true for all $m < N$. We prove the result holds for $m = N$.

By induction, there exists y such that $E_q E_{s_1} E_{s_2} E_{s_3} \cdots E_{s_{m-1}} = c' E_y$ and $y = qy'$ reduced. Define w' by $E_{w'} = cE_y E_{s_m}$.

There are three cases to consider. Either (1) $\ell(ys_m) < \ell(y)$, (2) $\ell(ys_m) > \ell(y)$ and $ys_m \in W_c$, or (3) $ys_m \notin W_c$.

In case 1, $c = [2]$ and $w' = y$, and the lemma holds.

In case 2, $c = 1$ and $w' = ys_m$, and the lemma holds.

In case 3, let $w_1 s_m t w_2$ be a decomposition of y as in property R. We may further assume that $w_1 s_m = qx'$, reduced, by commuting to the left of $s_m t$ all possible $s \in \mathrm{supp}(w_2)$. We have $E_{w_1} E_{s_m} E_t E_{w_2} E_{s_m} = E_{w_1} E_{s_m} E_{w_2}$. Although the product $w_1 s_m w_2$ is not necessarily reduced, it represents a product of generators of length less than $N + \ell(q)$. Thus, the lemma follows by induction. $\square$

Lemma 3.2.4. *Let $w = xq$, reduced, with $q \in Q$ and $a(w) = a(q)$. Assume that no element of $\mathcal{R}(w)$ is right cancellable. Let $s_1, s_2, s_3, \ldots, s_j$ be elements of S. Define $w' \in W_c$ by $cE_{w'} = E_w E_{s_1} E_{s_2} E_{s_3} \cdots E_{s_j}$, $c \in \mathcal{A}$. If $a(w') = a(q)$, then there exists $y \in W_c$ such that $w' = xqy$ reduced.*

Proof. We prove this by induction on j, the case where $j = 0$ being clear.

Suppose the lemma is true for $j < N$. Let $z \in W_c$ be defined by $c' E_z = E_w E_{s_1} E_{s_2} E_{s_3} \cdots E_{s_{N-1}}$, $c' \in \mathcal{A}$. If $a(z) \neq a(q)$, then $a(z) > a(q)$ and by corollary 3.2.2, this would imply that $a(w') > a(q)$, and there would be nothing to prove.

So assume that $a(z) = a(q)$. Then by induction, there exists $y' \in W_c$ such that $z = x\iota(U)y'$ reduced. Define u by $dE_u = E_z E_{s_N}$, $d \in \mathcal{A}$. It suffices to show that $u = x\iota(U)y''$ reduced, for some $y'' \in W_c$.

To see this, we consider the standard cases. If $zs_N \in W_c$, then $u = z$ or $u = zs_N$ depending on whether $\ell(zs_N) < \ell(z)$ or $\ell(zs_N) > \ell(z)$, respectively. In either case, there is nothing to show.

So assume $zs_N \notin W_c$. In this case, write $z = x_1 s_N t x_3$ as in property R. Observe that $t \notin \mathcal{R}(q)$ since by assumption, no element of $\mathcal{R}(w) = \mathcal{R}(q)$ is right cancellable. On the other hand, if $t \in \text{supp}(x)$ and $t \notin \text{supp}(y')$, we must have, by property R and maximality of $\#U$, that s_N commutes with all generators in U. Consequently $a(w') > a(q)$, and there is nothing to prove. The only remaining possibility is that $t \in \text{supp}(y')$ so that $E_z E_{s_N} = E_{x_1} E_{s_N} E_{x_3} = E_{x\iota(U)} E_{y'} E_{s_N}$, where $E_{y'} E_{s_N}$ can be written as a product of $\ell(y') - 1$ algebra generators E_s. We can therefore apply induction and the lemma follows. $\square$

3.3 Two-sided cells.

We now classify the two-sided cells $\mathbf{T}(w)$ which were defined in 2.2.

Theorem 3.3.1. *The two-sided cells of W_c are parametrized by $Q/\sim$ (see 3.1). The bijection is given by $q \mapsto \mathbf{T}(q)$. Furthermore, the a-function is constant on two-sided cells.*

Proof. We begin with the following lemma.

Lemma 3.3.2. *Let q, $q' \in Q$. We have $q \sim q'$ (see 3.1) if and only if $q \sim_{LR} q'$.*

Proof. First we show that $q \sim q'$ implies $q \sim_{LR} q'$. It suffices to check this in the case where q and q' are neighbours.

By definition, there exists s, $t \in S$ so that $E_q = E_s E_{q'} E_s$ and $E_{q'} = E_t E_q E_t$. This shows that $q \sim_{LR} q'$.

For the other implication, suppose $q \not\sim q'$. By lemma 3.2.1, every w such that there exists a constant c for which $cE_w = E_x E_q E_y$ must have a reduced expression which involves a substring equal to some q'' such that $q'' \sim q$. If $q \sim_{LR} q'$, then some substring of q is equal to q' and some substring of q' is equal to q. That is, $q = q'$, contrary to hypothesis. Thus, $q \not\sim_{LR} q'$. $\square$

Theorem 3.3.1 follows from lemma 3.3.2 and lemma 3.1.1. $\square$

3.4 Involutions in W_c.

Before we classify the right cells, we first study involutions in W_c.

Theorem 3.4.1. *Let $w \in W_c$ satisfy $w^2 = 1$. There exists a unique $T \in P$ and a unique $x \in W_c$ such that $w = x\iota(T)x^{-1}$ reduced.*

Because of the uniqueness of x and T in theorem 3.4.1, we shall call $x\iota(T)x^{-1}$ the canonical decomposition of an involution.

Proof. The proof is the same as the proof of theorem 3.1.1 in [**2**]. $\square$

Recall from the proof of theorem 3.1.1 in [**2**] that one can construct a finite sequence of triples (w_m, x_m, S_m) for $0 \leq m \leq N$, such that w_m, $x_m \in W_c$, $w_m^2 = 1$, $w = x_m w_m x_m^{-1}$ reduced, $S_m = \text{supp}(w_m)$, $\ell(x_{m+1}) = \ell(x_m) + 1$, and N is minimal such that $S_N \in P$.

In type affine A, we also note that $\text{supp}(w_m)$ is either equal to $\text{supp}(w_{m+1})$ or differs by a single node. Also, because no element of W_c whose support is a subdiagram of a graph of type A involves two occurrences of an end generator, we conclude that the number of connected components of $\text{supp}(w_m)$ is exactly 1 less than the number of connected components of $\text{supp}(w_{m+1})$ when $\text{supp}(w_{m+1}) \neq \text{supp}(w_m)$ and $\text{supp}(w_m) \neq \Gamma$. It follows that none of the graphs $\text{supp}(w_m)$ can contain a subgraph of type A_2 (since the two elements of W_c for type A_2 whose support is all of A_2 are not involutions). Thus, we have the following corollary.

Corollary 3.4.2. *When n is odd, no involution $d \in W_c$ has* $\text{supp}(d) = \Gamma$. *If* $\text{supp}(d) = \Gamma$, *then T is maximal with respect to inclusion in P, where $d = x\iota(T)x^{-1}$ is the canonical decomposition.*

Lemma 2.4.1 implies the following.

Corollary 3.4.3. *Let $d \in W_c$ be an involution and let $d = x\iota(T)x^{-1}$ be the canonical decomposition. Then no element of $\mathcal{R}(x\iota(T)) = T$ can be right cancelled.*

Proposition 3.4.4. *Let $d \in W_c$ be an involution and let $d = x\iota(T)x^{-1}$ be the canonical decomposition. Assume that* $\mathrm{supp}(d) \neq \Gamma$*. Then $d \sim_R x\iota(T)$, $d \sim_L \iota(T)x^{-1}$, and $d \sim_{LR} \iota(T)$.*

Proof. We assume $x \neq 1$, otherwise the statement is clear.

Consider the element $x\iota(T)$. Note that $\mathcal{R}(x\iota(T)) = T$ since $x\iota(T)x^{-1}$ is reduced. Note that no element of $\mathcal{R}(x\iota(T))$ is right cancellable since $x\iota(T)x^{-1} \in W_c$.

By proposition 3.1.2, we conclude that some element of $\mathcal{L}(x\iota(T))$ is left cancellable.

Using induction on the length of x, we conclude that $x\iota(T) \sim_L \iota(T)$. A similar argument shows that $\iota(T) \sim_R \iota(T)x^{-1}$. Thus, $x\iota(T) \sim_R x\iota(T)x^{-1}$ and $\iota(T)x^{-1} \sim_L x\iota(T)x^{-1}$. The proposition follows. □

Corollary 3.4.5. *Let $d \in W_c$ be an involution and let $d = x\iota(T)x^{-1}$ be the canonical decomposition. We have $a(d) = \#T$.*

Proof. We have $a(d) \geq \#T$. If $\#T$ is maximal, there is nothing to prove. Otherwise, by corollary 3.4.2, $\mathrm{supp}(d) \neq \Gamma$. By proposition 3.4.4 and the fact that the a-function is constant on two-sided cells, we see that $a(d) = a(\iota(T)) = \#T$. □

3.5 Right cells.

We are now ready to begin classifying right cells.

Theorem 3.5.1. *The map given by $d \mapsto \mathbf{R}(d)$, defines an injection from the set of involutions to the set of right cells. If n is odd, this map is surjective. If n is even, the right cells not in the image of this map are the right cells in the two-sided cells parametrized by elements of M which are not involutions.*

The remainder of this section is devoted to a proof of theorem 3.5.1.

First note that, by definition, each two-sided cell is a disjoint union of right cells. Let $q \in Q$. We shall first consider the case where $q \notin M$.

Let $w \in \mathbf{T}(q)$. By right cancelling repeatedly, we can find x such that $w \sim_R x$ and no element of $\mathcal{R}(x)$ can be right cancelled.

By lemma 3.2.1, we can write $x = yq'z$ where $q' \sim q$ and $a(x) = a(w)$. Let $\iota(G_1) \cdots \iota(G_m)$ be the right decomposition of x. Note that every element of G_{m-1}

must be connected to two elements of G_m, because no element of $\mathbf{R}(x)$ can be right cancelled. By lemma 2.4.1 and induction, we see that each element of G_k is connected to two elements of G_{k+1}. Since $\#\mathcal{R}(x) \neq n/2$, we see that the cardinality of G_k strictly increases with k. In other words, $q' = \iota(G_m)$ and $x = yq'$.

Observe that $yq'y^{-1}$ is reduced and an involution (compare lemma 4.3.1 of [**2**]). By proposition 3.4.4, we have $yq'y^{-1} \sim_R w$. Thus, every $w \in \mathbf{T}(q)$ is right equivalent to some involution. By lemma 3.2.4 and theorem 3.4.1, we deduce that every right cell in $\mathbf{T}(q)$ contains a unique involution.

Now assume that $q \in M$. Then n is even.

First, consider the element M_1. By proposition 3.1.4 and lemma 3.2.3, every element of $\mathbf{R}(M_1)$ is of the form $M_1 y$ reduced. If $M_1 y \sim_R M_1$ then $\mathcal{L}(y)$ must be a proper subset of M_2. By a similar argument used in the case where $q \notin M$, we conclude that any y such that $\mathcal{L}(y) \neq M_2$ satisfies $M_1 \sim_R M_1 y$.

These observations enable us to deduce that the right cells in the two-sided cell $\mathbf{T}(q)$ are parametrized by z such that $z\iota(\mathcal{L}(q)) \in W_c$ reduced and $\mathcal{R}(z) \subsetneqq S \setminus \mathcal{L}(q)$. We have

$$\mathbf{R}(zq) = \{zqy \mid \mathcal{L}(y) \subsetneqq S \setminus \mathcal{R}(q)\}.$$

If $\mathcal{R}(q) = \mathcal{L}(q)$, then $zqz^{-1} \in W_c$ is reduced and an involution.

The theorem follows. $\square$

4. A Diagram Calculus for $TL(\widehat{A}_l)$

Label the elements of S by $s_1, \ldots, s_n$, where s_i and s_j are connected in Γ whenever i and j are consecutive modulo n.

4.1 Affine n-diagrams.

Definition 4.1.1. An affine n-diagram consists of two infinite horizontal rows of nodes together with edges which can be made to satisfy the following conditions:

(i) Every node is the endpoint of exactly one edge.

(ii) Any edge lies entirely between the two rows of nodes.

(iii) If an edge does not link two nodes then it is an infinite horizontal line which does not meet any node.

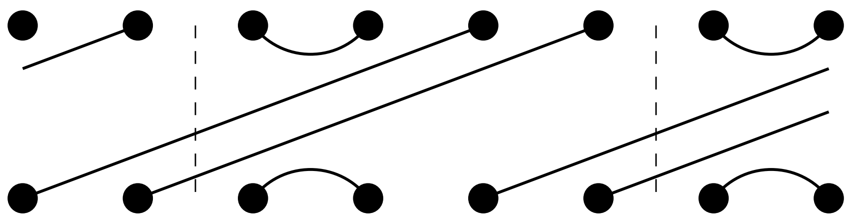
(iv) No two edges intersect each other.

(v) The diagram is invariant under shifting to the left or to the right by n nodes.

Because of the condition (v), one can also think of affine n-diagrams as diagrams on the surface of a cylinder, or within an annulus, in a natural way. Unless otherwise specified, we shall henceforth regard the diagrams as diagrams on the surface of a cylinder with n nodes on top and n nodes on the bottom. Compare with Jones' (finite-dimensional) annular algebra in [**4**].

An example of an affine n-diagram for $n = 4$ is given in figure 1.

FIGURE 1: AN AFFINE 4-DIAGRAM



4.2 Representation of $TL(\widehat{A}_l)$ by affine n-diagrams.

Consider the free $\mathcal{A}$-module with basis given by the affine n-diagrams. We construct an algebra structure on this $\mathcal{A}$-module by defining a multiplication as follows. For two diagrams A and B, put the cylinder for A on top of the cylinder for B so that corresponding vertices match and then identify all the points in the middle row. This produces a certain number, say x, of loops. Removal of these loops forms another diagram C satisfying the conditions in Definition 4.1.1. The product AB is then defined to be $[2]^x C$. One can check that this extends to an associative multiplication with unit. The unit is given by the diagram for which node $\bar{j}$ in the top ring is joined to node $\bar{j}$ in the bottom ring for all $\bar{j}$.

This definition generalizes the realization of the finite Temperley–Lieb algebra in terms of r-diagrams—see for example [**10**, §1].

Denote by E'_{s_i} the unique diagram in which $\bar{i}$ and $\overline{i+1}$ are joined in both rows of nodes, and for other congruence classes $\bar{j}$, node $\bar{j}$ in the top ring is joined to node $\bar{j}$ in the bottom ring.

Proposition 4.2.1. *There is a unique $\mathcal{A}$-algebra homomorphism ρ from $TL(\widehat{A}_l)$ to the algebra of affine n-diagrams which sends E_{s_i} to E'_{s_i}.*

Proof. One checks that the defining relations hold. □

The aim of the rest of §4 is to prove that ρ is injective and to characterize a set of diagrams which constitute a basis for $\rho(TL(\widehat{A}_l))$.

4.3 Monomials and diagrams.

Proposition 4.2.1 allows us to make the following definition.

Definition 4.3.1. Choose an element $w \in W_c$ of length k and fix a reduced expression $s_{i_1} \cdots s_{i_k}$ for w. The stacked representation of this reduced expression is the diagram, denoted E'_w, formed by stacking the diagrams E'_{s_i} on top of each other, with s_{i_1} at the top.

Lemma 4.3.2. *For $w \in W_c$, the diagram E'_w is an affine n-diagram.*

Proof. One has to check that the conditions of definition 4.1.1 are satisfied. All are immediate except (iii). For (iii), note that any edge either links two nodes or is equivalent to a closed loop on the cylinder. □

Remark 4.3.3. Let $w \in W_c$. Note that by construction, $\rho(E_w) = E'_w$.

Lemma 4.3.4. *Let $w \in W_c$. Then we have:*

(i) *Points $\bar{i}$ and $\overline{i+1}$ in the top ring of E'_w are connected to each other by an edge of minimal length if and only if $\ell(s_i w) = \ell(w) - 1$.*

(ii) *Points $\bar{i}$ and $\overline{i+1}$ in the bottom ring of E'_w are connected to each other by an edge of minimal length if and only if $\ell(w s_i) = \ell(w) - 1$.*

Proof. Part (i) follows from the fact that $\ell(s_i w) < \ell(w)$ if and only if $w = s_i w'$ reduced, and from the definition of E'_w. Part (ii) follows from the fact that $\ell(w s_i) < \ell(w)$ if and only if $w = w' s_i$ reduced, and from the definition of E'_w. □

Note that the number of times s_k appears in some reduced expression for $w \in W_c$ is independent of the reduced expression for w. Denote this number by $\nu(k, w)$.

We shall have to consider auxiliary vertical lines on the cylinder. A vertical line which passes between nodes $\bar{k}$ and $\overline{k+1}$ will be called the vertical line $k + 1/2$.

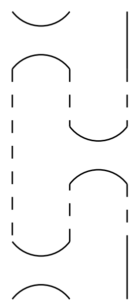
Lemma 4.3.5.

(i) *Let k be an integer from 1 to n inclusive. Then the number of intersections of E'_w with the vertical line $k + 1/2$ is equal to $2\nu(k, w)$.*

(ii) *Let $w \in W_c$. Then the path traced out by an edge in the diagram E'_w either has no segments moving to the left, or no segments moving to the right.*

Note. Informally, (ii) is saying that the edges in a diagram corresponding to an element of W_c do not double back on themselves.

FIGURE 2: A DIAGRAMMATIC VERSION OF PROPERTY R



Proof. We first show (i). We prove this by induction on the length of w, the case where $\ell(w)$ is 0 or 1 being clear.

Suppose the lemma holds for w such that $\ell(w) < l$, where $l > 1$. Let $w \in W_c$ have length l. We can write $w = w's_i$ reduced for some generator s_i. By induction, the number of intersections of $E'_{w'}$ with the vertical line $k+1/2$ is equal to $2\nu(k, w')$. There are two cases: either $i = k$ or $i \neq k$.

Suppose $i \neq k$. Then by definition of E'_{s_i} and E'_w, we see that the number of intersections of E'_w with the vertical line $k + 1/2$ is equal to the number of intersections of $E'_{w'}$ with the vertical line $k + 1/2$ and the lemma holds for w.

Suppose $i = k$. By lemma 4.3.4, we know that nodes $\bar{k}$ and $\overline{k+1}$ are not connected by an edge of minimal length so that no loops are formed in the diagram multiplication of $E'_{w'}$ with E'_{s_k}. Therefore, the number of intersections of E'_w with the vertical line $k + 1/2$ is equal to the number of intersections of $E'_{w'}$ with the vertical line $k + 1/2$ plus 2, and the lemma holds for w.

We now prove (ii), again by induction on $\ell(w)$. The base cases are clear. A short consideration shows that a counterexample of minimal length must consist of a diagram containing a subdiagram like the one shown in figure 2, or its left-right mirror image. (The dashed lines denote lines of indeterminate length.) However, this shows that the corresponding element w contains a reduced subword sw_1tw_2s, where w_1 and w_2 are words in the generators, s and t are generators which do not commute with each other, and s commutes with w_1 and w_2. Thus w has a reduced expression containing the sequence sts, and does not lie in W_c.

The lemma follows. □

Remark 4.3.6. The situation in figure 2 is a manifestation of Property R.

4.4 Straightening diagrams.

Consider a diagram D. Perturb the edges as necessary so that edges never run along a vertical line $k + 1/2$. Fix an edge and a node $\bar{k}$. If the edge has endpoints, we can compute the number of crossings of this edge with the vertical line $k+1/2$ by counting crossings with multiplicity $+1$ or -1 depending on whether the crossing is anti-clockwise or clockwise, adding these intersections multiplicities up and taking absolute value. If the edge has no endpoints, then we can declare the number of crossings with the vertical line $k + 1/2$ to be $+1$. Note that these numbers are only dependent upon the graph theoretic properties of D. We define $\nu_k(D)$ to be the sum of these (nonnegative) crossing numbers over all edges. It follows from Lemma 4.3.5 that $\nu_k(D) = 2\nu(k, w)$ when $D = E'_w$ for $w \in W_c$.

It is convenient to classify the edges in an affine n-diagram in the following way.

Definition 4.4.1. We call an edge linking two points on the same side (i.e. top or bottom) of an affine n-diagram a short horizontal edge. We call an edge with no endpoints a long horizontal edge. We call the other edges vertical.

Definition 4.4.2. An affine n-diagram D is said to be admissible if (a) it has no horizontal edges and is the image of the identity element in $TL(\widehat{A}_l)$ or (b) it has horizontal edges and the number $\nu_k(D)$ of intersections of the diagram with any vertical line $k + 1/2$ is an even number.

If D is admissible, the nonnegative integer

$$\frac{1}{2}\sum_{k=1}^{n} \nu_k(D)$$

is called the length of D, denoted by $\ell(D)$.

Note that this definition implies that there are only a finite number of long horizontal edges in any admissible diagram.

Definition 4.4.3. The admissible diagram D is called straight if it is of form $E'_{\iota(S)}$, $S \in P$.

The next aim is to express a non-straight diagram E'_w in a definite way as a product $E'_{s_i} E'_{w'}$ where $\ell(E'_{w'}) = \ell(E'_w) - 1$.

Definition 4.4.4. Let D be an admissible but non-straight diagram represented between two infinite horizontal lines.

We say that a congruence class $\bar{k}$ mod n is of type 1T for D if points k and $k+1$ in the top row of D can be connected by a short horizontal edge of minimal length, and either (a) D has a short horizontal edge connecting points i and j in its top row, where $i < k$ and $j > k+1$ are integers or if (b) D has no edges as described in (a), but has a long horizontal edge.

We define a class of type 1B as above but using the bottom edge.

We say that $\overline{k-1}$ is of type 2T for D if points k and $k+1$ in the top row of D can be joined to each other by a short horizontal edge of minimal length and top node $k-1$ is connected to a bottom node j with $j > k+1$.

We say that a class $\overline{k-1}$ is of type 2B for D if points k and $k+1$ in the bottom row of D can be joined to each other by a short horizontal edge of minimal length and bottom node $k-1$ is connected to a top node j with $j > k+1$.

Example 4.4.5. Consider the case $n = 4$. Then 2 is a congruence class of type 1T and 1B for the diagram E'_w where $w = s_2 s_1 s_3 s_2$. Also 1 is a congruence class of type 2T for the diagram $E'_{w'}$ where $w' = s_2 s_1$.

Lemma 4.4.6. *Any admissible but non-straight diagram D has a congruence class $\bar{k}$ mod n of type 1T, 1B, 2T or 2B.*

Proof. Let us assume D has no congruence classes of type 1T or 1B.

Suppose D has a vertical edge. Then not all the edges can be vertical unless D is the image of the identity element (which would mean D was straight, contrary to assumption), because D is admissible. It now follows that D will have a congruence class of type 2T if it has a vertical edge with negative gradient, and a congruence class of type 2B if it has a vertical edge with positive gradient. One of these situations must arise under the hypotheses on D. (It should be noted that no admissible diagram can have a vertical edge of minimal length connecting a top node a to a bottom node $a \pm 1$.)

Suppose that D has no vertical edges, and no congruence classes of type 1T or 1B. It therefore consists entirely of short horizontal edges of minimal length. The only such possibilities are diagrams which are either straight, or inadmissible. $\square$

Definition 4.4.7. We say the congruence class i mod n is a distinguished congruence for D if one of the following holds:

(i) D has a congruence class i of type 1T.

(ii) D has no congruence class of type 1T, but has a congruence class i of type 1B.

(iii) D has no congruence class of type 1T or 1B but has a congruence class i of type 2T.

(iv) D has a no congruence class of type 1T, 1B or 2T, but has a congruence class i of type 2B.

Note that exactly one of these conditions will hold. We classify the congruence to be of type (i), (ii), (iii) or (iv) respectively depending on which condition holds.

The following lemma is the main result of this section.

Lemma 4.4.8 (Straightening Rule). *Let D be an admissible but not straight diagram and let i be a distinguished congruence for D.*

If i is of type (i) for D then there exists a unique admissible (possibly straight) diagram D' of length $\ell(D) - 1$ such that $E'_{s_i} D' = D$.

If i is of type (ii) for D then there exists a unique admissible diagram D' of length $\ell(D) - 1$ such that $D' E'_{s_i} = D$.

If i is of type (iii) for D then we define $D' = E'_{s_i} D$. Then D' is admissible, $\ell(D') = \ell(D) - 1$ and $D = E'_{s_{i+1}} D'$.

If i is of type (iv) for D then we define $D' = D E'_{s_i}$. Then D' is admissible, $\ell(D') = \ell(D) - 1$ and $D = D' E'_{s_{i+1}}$.

In each case, D' has the same number of short horizontal edges as D.

Proof. The proof of (i) is by an entirely routine exhaustive check, so we omit it. The proof of (ii) is essentially the same as the proof of (i). The proof of (iv) is analogous to the proof of (iii), so we concentrate on proving (iii).

In the case of (iii) one verifies using the product rule that $E'_{s_i}D$ is equal to another diagram D' satisfying $\nu_{i+1}(D') = \nu_{i+1}(D) - 1$ and $\nu_k(D') = \nu_k(D)$ for $k \neq i+1$. The claim about the length of D' follows. The claim that $D = E'_{s_{i+1}}D'$ is a consequence of Lemma 4.3.4, the product rule, and the relation $E_{s_{i+1}}E_{s_i}E_{s_{i+1}} = E_{s_{i+1}}$. The claim about the number of short horizontal edges also follows by direct inspection. □

4.5 Consequences of the Straightening Rule.

Corollary 4.5.1. *Any admissible diagram is of the form E'_w for some $w \in W_c$. Conversely any E'_w, $w \in W_c$, is admissible.*

Proof. Suppose D is a counterexample to the statement of minimal length. Then D is clearly not straight. Lemma 4.4.8 now shows D to be the product of a shorter admissible diagram and a generator E'_{s_i}. By hypothesis, D' is of form $E'_{w'}$ for $\ell(w') = \ell(w) - 1$. Since no [2] appears, the proof of Lemma 4.3.4 shows that D is of form E'_w where $w = s_iw'$ or $w = w's_i$ according as $D = E'_{s_i}D'$ or $D = D'E'_{s_i}$.

By Lemma 4.3.4 and Lemma 4.3.5, one can see easily that any E_w maps under ρ to an admissible diagram. □

This means the image of $TL(\widehat{A}_l)$ in the diagram algebra is precisely the span of the admissible diagrams.

Theorem 4.5.2. *The diagram representation is faithful.*

Proof. Recall that a basis for $TL(\widehat{A}_l)$ is given by $\{E_w\}_{w\in W_c}$. Let a, $b \in W_c$. By Remark 4.3.3, if ρ were not faithful, two monomials E_a and E_b would map to the same diagram D. By Lemma 4.3.4, ρ preserves length, so we conclude that a and b have the same length.

Now suppose we have a counterexample of minimal length. Suppose D is not straight. Then by Lemma 4.4.8 either (i) $D = E'_{s_i}D'$ where $D' = E'_{a'} = E'_{b'}$,

$a = s_i a'$, $b = s_i b'$ for some i or (ii) $D = D'E'_{s_i}$ where $D' = E'_{a'} = E'_{b'}$, $a = a's_i$, $b = b's_i$ for some i.

In either case, $E'_{a'} = E'_{b'}$ is a shorter counterexample. So D must be straight.

By Lemma 4.3.4, if $D = E'_w$ then a reduced expression for w must contain all the commuting reflections associated to it by the definition of it being straight. A quick analysis of $\nu(k, w)$ for the various k using Lemma 4.3.5 now shows that w is unique. The proof follows. $\square$

We conclude by giving the relation between cell structure and diagrams.

Proposition 4.5.3. *Let $w \in W_c$. Then $a(w)$ is equal to the number of short horizontal edges on the top (or bottom) face.*

The two-sided cell in which w lies is determined by $a(w)$ and the number of long horizontal edges.

The left cell in which w lies is determined by its two-sided cell and the pattern of short horizontal edges on the bottom row.

The right cell in which w lies is determined by its two-sided cell and the pattern of short horizontal edges on the top row.

Proof. This follows by noting that left equivalence preserves the pattern of short horizontal edges on the bottom row, right equivalence preserves the pattern of short horizontal edges on the top row, and therefore, in both cases, the number of short horizontal edges is invariant. Since, by the results of section 3, for each $w \in W_c$ there exists $q \in Q$ such that $w \sim_{LR} q$, and because for $q \in Q$, we can see that $a(q)$ is the number of short horizontal edges in E'_q, the proposition follows. $\square$

Now using standard combinatorial arguments we have the following.

Corollary 4.5.4. *The number of left cells (or right cells) in a two-sided cell with a-value k, $k < n/2$, is equal to $\binom{n}{k}$. If $k = n/2$, then the number of left cells (or right cells) in a two-sided cell with a-value k is $\frac{1}{2}\binom{n}{k}$.*

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