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Presenting affine q -Schur algebras

Received: date / Revised: date

Abstract We obtain a presentation of certain affine q -Schur algebras in terms of generators and relations. The presentation is obtained by adding more relations to the usual presentation of the quantized enveloping algebra of type affine \mathfrak{gl}_n . Our results extend and rely on the corresponding result for the q -Schur algebra of the symmetric group, which were proved by the first author and Giaquinto.

Mathematics Subject Classification (2000) 17B37, 20F55

Introduction

Let V' be a vector space of finite dimension n . On the tensor space $(V')^{\otimes r}$ we have natural commuting actions of the general linear group $\mathrm{GL}(V')$ and the symmetric group \mathcal{S}_r . Schur observed that the centralizer algebra of each action equals the image of the other action in $\mathrm{End}((V')^{\otimes r})$, in characteristic zero, and Schur and Weyl used this observation to transfer information about the representations of \mathcal{S}_r to information about the representations of $\mathrm{GL}(V')$. That this Schur–Weyl duality holds in arbitrary characteristic was first observed in [4], although a special case was already used in [2]. In recent years, there have appeared various applications of the Schur–Weyl duality viewpoint to modular representations. The Schur algebras $S(n, r)$ first defined in [9] play a fundamental role in such interactions.

Jimbo [13] and (independently) Dipper and James [6] observed that the tensor space $(V')^{\otimes r}$ has a q -analogue in which the mutually centralizing actions of $\mathrm{GL}(V')$ and \mathcal{S}_r become mutually centralizing actions of a quantized enveloping algebra $\mathbf{U}(\mathfrak{gl}_n)$ and of the Iwahori–Hecke algebra $\mathcal{H}(\mathcal{S}_r)$ corresponding to \mathcal{S}_r . In

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this context, the ordinary Schur algebra $S(n, r)$ is replaced by the q -Schur algebra $\mathbf{S}_q(n, r)$. Dipper and James also showed that the q -Schur algebras determine the representations of finite general linear groups in non-defining characteristic.

An affine version of Schur–Weyl duality was first described in [3]. A different version, in which the vector space V' is replaced by an infinite dimensional vector space V , is given in [11], and we follow the latter approach here. In the affine (type A) setting, the mutually commuting actions are of an affine quantized enveloping algebra $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ and an extended affine Hecke algebra $\mathcal{H}(\widehat{W})$ corresponding to an extended affine Weyl group \widehat{W} containing the affine Weyl group W of type \widehat{A}_{r-1} . The affine q -Schur algebra $\widehat{\mathbf{S}}_q(n, r)$ in this context, which is also infinite dimensional, was first studied in [11], [17], and [20].

Recently, a new approach to Schur algebras or their q -analogues was given in [7], where it was shown that they may be defined by generators and relations in a manner compatible with the usual defining presentation of the enveloping algebra or its corresponding quantized enveloping algebra. The purpose of this paper is to extend that result to the affine case — that is, to describe the affine q -Schur algebra $\widehat{\mathbf{S}}_q(n, r)$ by generators and relations compatible with the defining presentation of $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$. This result is formulated in Theorem 1.6.1, under the assumption that $n > r$. An equivalent result, which describes the affine q -Schur algebra as a quotient of Lusztig’s modified form of the quantized enveloping algebra, is given in Theorem 2.6.1. These results depend on a different presentation, also valid for $n > r$, of the q -Schur algebra given in [11, Proposition 2.5.1]. A different approach to the results of this paper seems to be indicated for the case $n \leq r$.

The organization of the paper is as follows. In Section 1 we give necessary background information, and formulate our main result. In Section 2 we give the proof of Theorem 1.6.1, and we also give, in Section 2.6, the alternative presentation mentioned above. Finally, in Section 3 we outline the analogous results in the classical case, when the quantum parameter is specialized to 1.

After we submitted this paper, McGerty informed us that he has independently proved Theorem 2.6.1 using different methods; see [19].

1 Preliminaries and statement of main results

Our main result, stated in §1.6, is a presentation by generators and relations of the affine q -Schur algebra. In order to put this result in context, we review some of the definitions of the algebra that have been given in the literature.

1.1 Affine Weyl groups of type A

The affine Weyl group will play a key role, both in our definitions and our methods of proof, so we define it first.

The Weyl group we consider in this paper is that of type \widehat{A}_{r-1} , where we intend $r \geq 3$. This corresponds to the Dynkin diagram in Figure 1.1.1.

The number of vertices in the graph in Figure 1.1.1 is r , as the top vertex (numbered r) is regarded as an extra relative to the remainder of the graph, which is a Coxeter graph of type A_{r-1} .

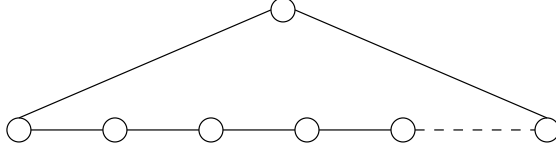


Fig. 1.1.1 Dynkin diagram of type \widehat{A}_{r-1}

We associate a Weyl group, $W = W(\widehat{A}_{r-1})$, to this Dynkin diagram in the usual way (as in [12, §2.1]). This associates to node i of the graph a generating involution s_i of W , where $s_i s_j = s_j s_i$ if i and j are not connected in the graph, and

$$s_i s_j s_i = s_j s_i s_j$$

if i and j are connected in the graph. For $t \in \mathbb{Z}$, it is convenient to denote by \bar{t} the congruence class of t modulo r , taking values in the set $\{1, 2, \dots, r\}$. For the purposes of this paper, it is helpful to think of the group W as follows, based on a result of Lusztig [15]. (Note that we write maps on the right when dealing with permutations.)

Proposition 1.1.2 *There exists a group isomorphism from W to the set of permutations of \mathbb{Z} satisfying the following conditions:*

$$(i+r)w = (i)w + r \tag{a}$$

$$\sum_{t=1}^r (t)w = \sum_{t=1}^r t \tag{b}$$

such that s_i is mapped to the permutation

$$t \mapsto \begin{cases} t & \text{if } \bar{t} \neq \bar{i}, \overline{i+1}, \\ t-1 & \text{if } \bar{t} = \overline{i+1}, \\ t+1 & \text{if } \bar{t} = \bar{i}, \end{cases}$$

for all $t \in \mathbb{Z}$.

For reasons relating to weight spaces which will become clear later, we consider a larger group \widehat{W} of permutations of \mathbb{Z} .

Definition 1.1.3 Let ρ be the permutation of \mathbb{Z} taking t to $t+1$ for all t . Then the group \widehat{W} is defined to be the subgroup of permutations of \mathbb{Z} generated by the group W and ρ .

As will become clear later, the point of ρ is that conjugation by ρ will correspond to a graph automorphism of the Dynkin diagram given by rotation by one place.

Proposition 1.1.4 (i) *There exists a group isomorphism from \widehat{W} to the set of permutations of \mathbb{Z} satisfying the following conditions:*

$$(i+r)w = (i)w + r \tag{a}$$

$$\sum_{t=1}^r (t)w \equiv \sum_{t=1}^r t \pmod{r}. \quad (\text{b})$$

(ii) Any element of \widehat{W} is uniquely expressible in the form $\rho^z w$ for $z \in \mathbb{Z}$ and $w \in W$. Conversely, any element of this form is an element of \widehat{W} .

(iii) Let $S \cong \mathcal{S}_r$ be the subgroup of \widehat{W} generated by

$$\{s_1, s_2, \dots, s_{r-1}\}.$$

Let Z be the subgroup of \widehat{W} consisting of all permutations z satisfying

$$(t)z \equiv t \pmod{r}$$

for all t . Then $\mathbb{Z}^r \cong Z$, Z is normal in \widehat{W} , and \widehat{W} is the semidirect product of S and Z .

Proof The three parts are proved in [11, Proposition 1.1.3, Corollary 1.1.4, Proposition 1.1.5] respectively. \square

It is convenient to extend the usual notion of the length of an element of a Coxeter group to the group \widehat{W} in the following way.

Definition 1.1.5 For $w \in W$ the length $\ell(w)$ of w is the length of a word of minimal length in the group generators s_i of W which is equal to w . The length, $\ell(w')$, of a typical element $w' = \rho^z w$ of \widehat{W} (where $z \in \mathbb{Z}$ and $w \in W$) is defined to be $\ell(w)$.

When the affine Weyl group is thought of in the above way, the familiar notions of length and distinguished coset representatives may be adapted from the corresponding notions for Coxeter groups.

Definition 1.1.6 Let Π be the set of subsets of $S = \{s_1, s_2, \dots, s_r\}$, excluding S itself. For each $\pi \in \Pi$, we define the subgroup \widehat{W}_π of \widehat{W} to be that generated by $\{s_i \in \pi\}$. (Such a subgroup is called a parabolic subgroup.) We will sometimes write W_π for \widehat{W}_π to emphasize that it is a subgroup of W . Let Π' be the set of elements of Π that omit the generator s_r .

All the subgroups \widehat{W}_π are subgroups of W , and are parabolic subgroups in the usual sense of Coxeter groups. Furthermore, each such \widehat{W}_π is isomorphic to a direct product of Coxeter groups of type A (i.e., finite symmetric groups) corresponding to the connected components of the Dynkin diagram obtained after omitting the elements s_i that do not occur in π . We will appeal to these facts freely in the sequel.

Definition 1.1.7 Let $\pi \in \Pi$. The subset \mathcal{D}_π of \widehat{W} is the set of those elements such that for any $w \in \widehat{W}_\pi$ and $d \in \mathcal{D}_\pi$,

$$\ell(wd) = \ell(w) + \ell(d).$$

We call \mathcal{D}_π the set of distinguished right coset representatives of \widehat{W}_π in \widehat{W} .

The subset \mathcal{D}_π^{-1} is called the set of distinguished left coset representatives of \widehat{W}_π in \widehat{W} ; elements $d \in \mathcal{D}_\pi^{-1}$ have the property that $\ell(dw) = \ell(d) + \ell(w)$ for any $w \in \widehat{W}_\pi$.

Proposition 1.1.8 (i) Let $\pi \in \Pi$ and $w \in \widehat{W}$. Then $w = w_\pi w^\pi$ for a unique $w_\pi \in \widehat{W}_\pi$ and $w^\pi \in \mathcal{D}_\pi$.
(ii) Let $\pi' \in \Pi$ and $w \in \widehat{W}$. Then $w = w^{\pi'} w_{\pi'}$ for a unique $w_{\pi'} \in \widehat{W}_{\pi'}$ and $w^{\pi'} \in \mathcal{D}_{\pi'}$.
(iii) Let $\pi_1, \pi_2 \in \Pi$. The set $\mathcal{D}_{\pi_1, \pi_2} := \mathcal{D}_{\pi_1} \cap \mathcal{D}_{\pi_2}^{-1}$ is an irredundantly described set of double $\widehat{W}_{\pi_1} \widehat{W}_{\pi_2}$ -coset representatives, each of minimal length in its double coset.

Proof See [11, Propositions 1.4.4, 1.4.5]. \square

1.2 Affine Hecke algebras of type A

We now define the extended affine Hecke algebra $\mathcal{H} = \mathcal{H}(\widehat{W})$ of type A. The Hecke algebra is a q -analogue of the group algebra of \widehat{W} , and is related to \widehat{W} in the same way as the Hecke algebra $\mathcal{H}(\mathcal{S}_r)$ of type A is related to the symmetric group \mathcal{S}_r . In particular, one can recover the group algebra of \widehat{W} by replacing the parameter q occurring in the definition of $\mathcal{H}(\widehat{W})$ by 1.

Definition 1.2.1 The affine Hecke algebra $\mathcal{H} = \mathcal{H}(\widehat{W})$ over $\mathbb{Z}[q, q^{-1}]$ is the associative, unital algebra with algebra generators

$$\{T_{s_1}, \dots, T_{s_r}\} \cup \{T_\rho, T_\rho^{-1}\}$$

and relations

$$T_s^2 = (q-1)T_s + q, \quad (1)$$

$$T_s T_t = T_t T_s \text{ if } s \text{ and } t \text{ are not adjacent in the Dynkin diagram,} \quad (2)$$

$$T_s T_t T_s = T_t T_s T_t \text{ if } s \text{ and } t \text{ are adjacent in the Dynkin diagram,} \quad (3)$$

$$T_\rho T_{s_{r+1}} T_\rho^{-1} = T_{s_1}. \quad (4)$$

In relation (4), we interpret s_{r+1} to mean s_1 .

The algebra \mathcal{H} has a better known presentation, known as the Bernstein presentation, but this is not convenient for our purposes. The equivalence of the two presentations is well known, and a proof may be found, for example, in [11, Theorem 4.2.5]. However, it will be convenient to have the following modified version of the presentation in Definition 1.2.1.

Lemma 1.2.2 The affine Hecke algebra $\mathcal{H} = \mathcal{H}(\widehat{W})$ over $\mathbb{Z}[q, q^{-1}]$ is the associative, unital algebra with algebra generators

$$\{T_{s_1}, \dots, T_{s_{r-1}}\} \cup \{T_\rho, T_\rho^{-1}\}$$

and relations

$$T_{s_i}^2 = (q-1)T_{s_i} + q, \quad (1')$$

$$T_{s_i} T_{s_j} = T_{s_j} T_{s_i} \text{ if } |i-j| > 1, \quad (2')$$

$$T_{s_i} T_{s_j} T_{s_i} = T_{s_j} T_{s_i} T_{s_j} \text{ if } |i-j| = 1, \quad (3')$$

$$T_\rho T_{s_{i+1}} T_\rho^{-1} = T_{s_i} \text{ if } 1 \leq i < r-1, \quad (4')$$

$$T_\rho^r T_{s_i} T_\rho^{-r} = T_{s_i} \text{ if } 1 \leq i \leq r-1, \quad (5')$$

Proof It is clear that relations (1')–(5') are consequences of relations (1)–(4). For the converse direction, we define $T_{s_r} := T_\rho T_{s_1} T_\rho^{-1}$; the remaining cases of relations (1)–(4) may then be obtained from relations (1')–(5') by conjugating by T_ρ or by T_ρ^{-1} . \square

Definition 1.2.3 Let $w \in W$. The element T_w of $\mathcal{H}(W)$ is defined as

$$T_{s_{i_1}} \cdots T_{s_{i_m}},$$

where $s_{i_1} \cdots s_{i_m}$ is a reduced expression for w (i.e., one with m minimal). (This is well-defined by standard properties of Coxeter groups.)

If $w' \in \widehat{W}$ is of form $\rho^z w$ for $w \in W$, we denote by $T_{w'}$ the element $T_\rho^z T_w$. (This is well-defined by Proposition 1.1.4 (ii).)

Proposition 1.2.4 (i) A free $\mathbb{Z}[q, q^{-1}]$ -basis for \mathcal{H} is given by the set $\{T_w : w \in \widehat{W}\}$.

(ii) As a $\mathbb{Z}[q, q^{-1}]$ -algebra, \mathcal{H} is generated by T_{s_1} , T_ρ and T_ρ^{-1} .

Proof See [11, Proposition 1.2.3, Lemma 1.2.4]. \square

1.3 The affine q -Schur algebra as an endomorphism algebra

We first present the definition of the affine q -Schur algebra as given in [11, §2].

Definition 1.3.1 A weight is a composition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of r into n pieces, that is, a finite sequence of nonnegative integers whose sum is r . (There is no monotonicity assumption on the sequence.) We denote the set of weights by $\Lambda(n, r)$.

The r -tuple $\ell(\lambda)$ of a weight λ is the weakly increasing sequence of integers where there are λ_i occurrences of the entry i .

The Young subgroup $\mathcal{S}_\lambda \subseteq \mathcal{S}_r \subseteq W \subseteq \widehat{W}$ is the subgroup of permutations of the set $\{1, 2, \dots, r\}$ that leaves invariant the following sets of integers:

$$\{1, 2, \dots, \lambda_1\}, \{\lambda_1 + 1, \lambda_1 + 2, \dots, \lambda_1 + \lambda_2\}, \{\lambda_1 + \lambda_2 + 1, \dots\}, \dots$$

The weight ω is given by the n -tuple

$$\underbrace{(1, 1, \dots, 1)}_r, \underbrace{(0, 0, \dots, 0)}_{n-r}.$$

Remark 1.3.2 The Young subgroup $\mathcal{S}_\lambda \subseteq \mathcal{S}_r$ can be thought of as a group \widehat{W}_λ for some $\lambda \in \Pi'$. Note, however, that different compositions λ can give rise to canonically isomorphic groups. Also note that we require $n \geq r$ for ω to exist.

Definition 1.3.3 Let $\lambda \in \Pi$. For $t \in \mathbb{Z}$, the parabolic subgroup $\widehat{W}_{\lambda+t}$ is the one generated by those elements s_{i+t}^{-1} where i is such that s_i lies in \widehat{W}_λ . We also use the notation $\mathcal{D}_{\lambda+t}$ with the obvious meaning.

The element $x_{\lambda+t} \in \mathcal{H}$ is defined as

$$x_{\lambda+t} := \sum_{w \in \widehat{W}_{\lambda+t}} T_w.$$

We will write x_λ for $x_{\lambda+0}$.

Definition 1.3.4 The affine q -Schur algebra $\widehat{\mathbf{S}}_q(n, r)$ over $\mathbb{Z}[q, q^{-1}]$ is defined by

$$\widehat{\mathbf{S}}_q(n, r) := \text{End}_{\mathcal{H}} \left(\bigoplus_{\lambda \in \Lambda(n, r)} x_\lambda \mathcal{H} \right),$$

where $\mathcal{H} = \mathcal{H}(\widehat{W})$.

There is a basis for $\widehat{\mathbf{S}}_q(n, r)$ similar to Dipper and James' basis for the ordinary q -Schur algebra.

Definition 1.3.5 Let $d \in \widehat{W}$ be an element of $\mathcal{D}_{\lambda, \mu}$. Write $d = \rho^z c$ (as in Proposition 1.1.4 (ii)) with $c \in W$. Then the element

$$\phi_{\lambda, \mu}^d \in \text{Hom}(x_\mu \mathcal{H}(\widehat{W}), x_\lambda \mathcal{H}(\widehat{W}))$$

is defined as

$$\begin{aligned} \phi_{\lambda, \mu}^d(x_\mu) &:= \sum_{d' \in \mathcal{D}_v \cap W_\mu} x_\lambda T_\rho^z T_{cd'} \\ &= \sum_{d' \in \mathcal{D}_v \cap W_\mu} T_\rho^z x_{\lambda+z} T_{cd'} = \sum_{w \in W_{\lambda+z} c W_\mu} T_\rho^z T_w = \sum_{w \in W_\lambda d W_\mu} T_w \end{aligned}$$

where v is the composition of n corresponding to the standard Young subgroup

$$d^{-1} W_{\lambda+z} d \cap W_\mu$$

of W .

Theorem 1.3.6 (i) A free $\mathbb{Z}[q, q^{-1}]$ -basis for $\widehat{\mathbf{S}}_q(n, r)$ is given by the set

$$\{\phi_{\lambda, \mu}^d : \lambda, \mu \in \Lambda(n, r), d \in \mathcal{D}_{\lambda, \mu}\}.$$

(ii) The set of basis elements

$$\{\phi_{\lambda, \mu}^d : \lambda, \mu \in \Lambda(n, r), d \in \mathcal{S}_r \cap \mathcal{D}_{\lambda, \mu}\}$$

spans a subalgebra of $\widehat{\mathbf{S}}_q(n, r)$ canonically isomorphic to the q -Schur algebra $\mathbf{S}_q(n, r)$.

(iii) *The set of basis elements*

$$\{\phi_{\omega,\omega}^d : d \in \widehat{W}\}$$

spans a subalgebra canonically isomorphic to the Hecke algebra $\mathcal{H}(\widehat{W})$, where $\phi_{\omega,\omega}^d$ is identified with T_d .

Proof See [11, Theorem 2.2.4] for part (i), and [11, Proposition 2.2.5] for parts (ii) and (iii). \square

Note again that parts (ii) and (iii) of Theorem 1.3.6 only apply if $n \geq r$.

1.4 Quantum groups and tensor space

The affine q -Schur algebras are closely related to certain quantum groups (Hopf algebras). The following Hopf algebra is crucial for our purposes.

Definition 1.4.1 The associative, unital algebra $\mathbf{U}(\mathfrak{gl}_n)$ over $\mathbb{Q}(v)$ is given by generators

$$E_i, F_i \quad (1 \leq i \leq n-1); \quad K_i, K_i^{-1} \quad (1 \leq i \leq n)$$

subject to the following relations:

$$K_i K_j = K_j K_i, \tag{Q1}$$

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \tag{Q2}$$

$$K_i E_j = v^{\varepsilon^+(i,j)} E_j K_i, \tag{Q3}$$

$$K_i F_j = v^{\varepsilon^-(i,j)} F_j K_i, \tag{Q4}$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{v - v^{-1}}, \tag{Q5}$$

$$E_i E_j = E_j E_i \quad \text{if } i \text{ and } j \text{ are not adjacent,} \tag{Q6}$$

$$F_i F_j = F_j F_i \quad \text{if } i \text{ and } j \text{ are not adjacent,} \tag{Q7}$$

$$E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad \text{if } i \text{ and } j \text{ are adjacent,} \tag{Q8}$$

$$F_j^2 F_i - (v + v^{-1}) F_j F_i F_j + F_i F_j^2 = 0 \quad \text{if } i \text{ and } j \text{ are adjacent.} \tag{Q9}$$

Here, we regard i and j as ‘‘adjacent’’ if i and j index adjacent nodes in the Dynkin diagram of type A_{n-1} . In the relations, i and j vary over all values of the indices for which the relation is defined. Also,

$$\varepsilon^+(i, j) := \begin{cases} 1 & \text{if } j = i; \\ -1 & \text{if } \bar{j} = \overline{i-1}; \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\varepsilon^-(i, j) := \begin{cases} 1 & \text{if } \bar{j} = \overline{i-1}; \\ -1 & \text{if } j = i; \\ 0 & \text{otherwise} \end{cases}$$

where we write \bar{a} for $a \in \mathbb{Z}$ to denote the residue class of a in the residue class ring $\mathbb{Z}/n\mathbb{Z}$. The residue class notation has no effect in the above definition, where indices are restricted to the range $1, \dots, n-1$. However, the notation is important in the next two definitions.

The following Hopf algebra is a quantized affine enveloping algebra associated with the affine Lie algebra $\widehat{\mathfrak{gl}}_n$.

Definition 1.4.2 The associative, unital algebra $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ over $\mathbb{Q}(v)$ is given by generators

$$E_i, F_i, K_i, K_i^{-1}$$

(where $1 \leq i \leq n$) subject to relations (Q1) to (Q9) of Definition 1.4.1 (reading indices modulo n).

In this definition, the notion of “adjacent” in relations (Q1)–(Q9) must now be interpreted in the Dynkin diagram of type \widehat{A}_{n-1} . More precisely, i and j are to be regarded as “adjacent” if i and j index adjacent nodes in the Dynkin diagram of type \widehat{A}_{n-1} . Note that i, j index adjacent nodes if and only if $i - j \equiv \pm 1 \pmod{n}$.

In [11], a larger Hopf algebra is considered. It is an extended version of the quantized affine algebra $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ considered in Definition 1.4.2.

Definition 1.4.3 The associative, unital algebra $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ over $\mathbb{Q}(v)$ is given by generators

$$E_i, F_i, K_i, K_i^{-1}, R, R^{-1}$$

(where $1 \leq i \leq n$) subject to relations (Q1) to (Q9) of Definition 1.4.1 (reading indices modulo n), together with the relations

$$RR^{-1} = R^{-1}R = 1, \tag{Q10}$$

$$R^{-1}K_{i+1}R = K_i, \tag{Q11}$$

$$R^{-1}K_{i+1}^{-1}R = K_i^{-1}, \tag{Q12}$$

$$R^{-1}E_{i+1}R = E_i, \tag{Q13}$$

$$R^{-1}F_{i+1}R = F_i. \tag{Q14}$$

The following result was proved in [11, Theorem 3.1.10].

Theorem 1.4.4 *The algebra $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ is a Hopf algebra with multiplication μ , unit η , comultiplication Δ , counit ε and antipode S . The comultiplication is defined by*

$$\Delta(1) = 1 \otimes 1,$$

$$\Delta(E_i) = E_i \otimes K_i K_{i+1}^{-1} + 1 \otimes E_i,$$

$$\Delta(F_i) = K_i^{-1} K_{i+1} \otimes F_i + F_i \otimes 1,$$

$$\Delta(X) = X \otimes X \text{ for } X \in \{K_i, K_i^{-1}, R, R^{-1}\}.$$

The counit is defined by

$$\begin{aligned}\varepsilon(E_i) &= \varepsilon(F_i) = 0, \\ \varepsilon(K_i) &= \varepsilon(K_i^{-1}) = \varepsilon(R) = \varepsilon(R^{-1}) = 1.\end{aligned}$$

The antipode is defined by

$$\begin{aligned}S(E_i) &= -E_i K_i^{-1} K_{i+1}, \\ S(F_i) &= -K_i K_{i+1}^{-1} F_i, \\ S(K_i) &= K_i^{-1}, \\ S(K_i^{-1}) &= K_i, \\ S(R) &= R^{-1}, \\ S(R^{-1}) &= R.\end{aligned}$$

The unit satisfies $\eta(1) = 1_U$.

Note that the usual Hopf algebra structure on $\mathbf{U}(\mathfrak{gl}_n)$ and $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ is obtained by restricting the operations of Theorem 1.4.4 above.

Let V be the $\mathbb{Q}(v)$ -vector space with basis $\{e_t : t \in \mathbb{Z}\}$. This has a natural $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ -module structure as follows.

Lemma 1.4.5 *There is a left action of $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ on V defined by the conditions*

$$\begin{aligned}E_i e_{t+1} &= e_t \text{ if } i = t \pmod{n}, \\ E_i e_{t+1} &= 0 \text{ if } i \neq t \pmod{n}, \\ F_i e_t &= e_{t+1} \text{ if } i = t \pmod{n}, \\ F_i e_t &= 0 \text{ if } i \neq t \pmod{n}, \\ K_i e_t &= v e_t \text{ if } i = t \pmod{n}, \\ K_i e_t &= e_t \text{ if } i \neq t \pmod{n}, \\ R e_t &= e_{t+1}.\end{aligned}$$

Proof See [11, Lemma 3.2.1]. □

Since $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ is a Hopf algebra, the tensor product of two $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ -modules has a natural $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ -module structure via the comultiplication Δ .

Definition 1.4.6 The vector space $V^{\otimes r}$ has a natural $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ -module structure given by

$$u \cdot x = \Delta(u)^{(r-1)} x.$$

We call this module *tensor space*. The weight $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n, r)$ of a basis element

$$e_{t_1} \otimes e_{t_2} \otimes \cdots \otimes e_{t_r}$$

of $V^{\otimes r}$ is given by the condition

$$\lambda_i := |\{j : t_j \equiv i \pmod{n}\}|$$

for $i = 1, \dots, n$. The λ -weight space, V_λ , of $V^{\otimes r}$ is the span of all the basis vectors of weight λ .

Henceforth, we will always assume that $q = v^2$, and regard $\mathbb{Q}(v)$ as an algebra over $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ by means of the ring homomorphism $\mathcal{A} \rightarrow \mathbb{Q}(v)$ such that $q \rightarrow v^2, q^{-1} \rightarrow v^{-2}$.

The following result about the affine q -Schur algebra, which will be used frequently in the sequel, was proved in [11, Theorem 3.4.8].

Theorem 1.4.7 *The quotient of $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ by the kernel of its action on tensor space is isomorphic as a $\mathbb{Q}(v)$ -algebra to the algebra $\mathbb{Q}(v) \otimes_{\mathcal{A}} \widehat{\mathbf{S}}_q(n, r)$.*

There is a corresponding result for the finite q -Schur algebra. This was introduced in [1]; see [8] or [10] for more details.

Theorem 1.4.8 *Let V' be the submodule of V spanned by the e_j for $1 \leq j \leq n$. Then the quotient of $\mathbf{U}(\mathfrak{gl}_n)$ by the kernel of its action on $(V')^{\otimes r}$ is isomorphic as a $\mathbb{Q}(v)$ -algebra to the algebra $\mathbb{Q}(v) \otimes_{\mathcal{A}} \mathbf{S}_q(n, r)$. We denote the corresponding epimorphism from $\mathbf{U}(\mathfrak{gl}_n)$ to $\mathbb{Q}(v) \otimes_{\mathcal{A}} \mathbf{S}_q(n, r)$ by α .*

Definition 1.4.9 For convenience of notation, we shall henceforth denote by $\widehat{\mathbf{S}}_v(n, r)$ the algebra $\mathbb{Q}(v) \otimes_{\mathcal{A}} \widehat{\mathbf{S}}_q(n, r)$ and by $\mathbf{S}_v(n, r)$ its finite analogue $\mathbb{Q}(v) \otimes_{\mathcal{A}} \mathbf{S}_q(n, r)$. We may refer to these algebras as the affine v -Schur algebra and v -Schur algebra, respectively.

It will be useful in the sequel to consider the weight spaces of $\mathbf{S}_v(n, r)$ as right $\mathbb{Q}(v) \otimes_{\mathcal{A}} \mathcal{H}(\mathcal{S}_r)$ modules. The following result is useful in such a context.

Lemma 1.4.10 *Let $1 \leq i_1 \leq i_2 \leq \dots \leq i_r \leq n$, and let $\lambda \in \Lambda(n, r)$ be such that λ_j is the number of occurrences of j in the sequence (i_1, i_2, \dots, i_r) . Then the λ -weight space of $V'^{\otimes r}$ is generated as a right $\mathbb{Q}(v) \otimes_{\mathcal{A}} \mathcal{H}(\mathcal{S}_r)$ -module by the element*

$$e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_r}.$$

Proof This is a well known result, which can be seen for example by using the definition of $\mathbf{S}_v(n, r)$ together with the isomorphism, given in [8], between tensor-space and Dipper and James' " q -tensor space" (see [6]). \square

Although $(V')^{\otimes r}$ is not a $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ -module, we have the following

Lemma 1.4.11 *The action of $u \in \widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ on $V^{\otimes r}$ is determined by its action on the subspace $(V')^{\otimes r}$.*

Proof This is part of [11, Proposition 3.2.5]. \square

1.5 Lusztig's approach

In §1.5 we review the approach to the affine q -Schur algebra used by Lusztig [17], McGerty [18] and others. We start by recalling McGerty's definitions from [18, §2].

Let V_ε be a free rank r module over $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$, where \mathbf{k} is a finite field of q elements, and ε is an indeterminate.

Let \mathcal{F}^n be the space of n -step periodic lattices, i.e. sequences $\mathbf{L} = (L_i)_{i \in \mathbb{Z}}$ of lattices in our free module V_ε such that $L_i \subset L_{i+1}$, and $L_{i-n} = \varepsilon L_i$. The group $G = \text{Aut}(V)$ acts on \mathcal{F}^n in the natural way. Let $\mathfrak{S}_{r,n}$ be the set of nonnegative integer sequences $(a_i)_{i \in \mathbb{Z}}$, such that $a_i = a_{i+n}$ and $\sum_{i=1}^n a_i = r$, and let $\mathfrak{S}_{r,n,n}$ be the set of $\mathbb{Z} \times \mathbb{Z}$ matrices $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ with nonnegative entries such that $a_{i,j} = a_{i+n,j+n}$ and $\sum_{i \in [1,n], j \in \mathbb{Z}} a_{i,j} = r$. The orbits of G on \mathcal{F}^n are indexed by $\mathfrak{S}_{r,n}$, where \mathbf{L} is in the orbit $\mathcal{F}_{\mathbf{a}}$ corresponding to \mathbf{a} if $a_i = \dim_{\mathbf{k}}(L_i/L_{i-1})$. The orbits of G on $\mathcal{F}^n \times \mathcal{F}^n$ are indexed by the matrices $\mathfrak{S}_{r,n,n}$, where a pair $(\mathbf{L}, \mathbf{L}')$ is in the orbit \mathcal{O}_A corresponding to A if

$$a_{i,j} = \dim \left(\frac{L_i \cap L'_j}{(L_{i-1} \cap L'_j) + (L_i \cap L'_{j-1})} \right).$$

For $A \in \mathfrak{S}_{r,n,n}$ let $r(A), c(A) \in \mathfrak{S}_{r,n}$ be given by $r(A)_i = \sum_{j \in \mathbb{Z}} a_{i,j}$ and $r(A)_j = \sum_{i \in \mathbb{Z}} a_{i,j}$.

Similarly let \mathcal{B}^r be the space of complete periodic lattices, that is, sequences of lattices $\mathbf{L} = (L_i)$ such that $L_i \subset L_{i+1}$, $L_{i-r} = \varepsilon L_i$, and $\dim_{\mathbf{k}}(L_i/L_{i-1}) = 1$ for all $i \in \mathbb{Z}$. Let $\mathbf{b}_0 = (\dots, 1, 1, \dots)$. The orbits of G on $\mathcal{B}^r \times \mathcal{B}^r$ are indexed by matrices $A \in \mathfrak{S}_{n,n,n}$ where the matrix A must have $r(A) = c(A) = \mathbf{b}_0$.

Let $\mathfrak{A}_{r,q}$, $\mathfrak{H}_{r,q}$ and $\mathfrak{T}_{r,q}$ be the span of the characteristic functions of the G orbits on $\mathcal{F}^n \times \mathcal{F}^n$, $\mathcal{B}^r \times \mathcal{B}^r$ and $\mathcal{F}^n \times \mathcal{B}^r$ respectively. Convolution makes $\mathfrak{A}_{r,q}$ and $\mathfrak{H}_{r,q}$ into algebras and $\mathfrak{T}_{r,q}$ into a $\mathfrak{A}_{r,q}$ - $\mathfrak{H}_{r,q}$ bimodule. For $A \in \mathfrak{S}_{r,n,n}$ set

$$d_A = \sum_{i \geq k, j < l, 1 \leq i \leq n} a_{ij} a_{kl}.$$

Let $\{e_A : A \in \mathfrak{S}_{r,n,n}\}$ be the basis of $\mathfrak{A}_{r,q}$ given by the characteristic function of the orbit corresponding to A , and let $\{[A] : A \in \mathfrak{S}_{r,n,n}\}$ be the basis of $\mathfrak{A}_{r,q}$ given by $[A] = q^{-d_A/2} e_A$. When $n = r$, the subset of either basis spanned by all monomial matrices A spans $\mathfrak{H}_{r,q}$.

All of these spaces of functions are the specialization at $v = \sqrt{q}$ of modules over $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, which we denote by \mathfrak{A}_r , \mathfrak{H}_r and \mathfrak{T}_r respectively; here v is an indeterminate.

Proposition 1.5.1 (Varagnolo–Vasserot) *The \mathcal{A} -algebra \mathfrak{A}_r is naturally isomorphic to the affine q -Schur algebra $\widehat{\mathbf{S}}_v(n, r)$ of Definition 1.3.4. Furthermore, the isomorphism may be chosen to identify the basis of Definition 1.3.5 with the basis $\{e_A : A \in \mathfrak{S}_{r,n,n}\}$.*

Proof The necessary isomorphism is the map Φ given in [20, Proposition 7.4 (a)]. \square

We will also need the *canonical basis*, $\{\{A\} : A \in \mathfrak{S}_{r,n,n}\}$, for $\widehat{\mathbf{S}}_v(n, r)$. This is related to the basis $\{[A] : A \in \mathfrak{S}_{r,n,n}\}$ in a unitriangular way: we have

$$\{A\} = \sum_{A_1: A_1 \leq A} \Pi_{A_1, A} [A_1],$$

where \leq is a certain natural partial order and the $\Pi_{A_1, A}$ are certain Laurent polynomials (similar to the famous Kazhdan–Lusztig polynomials $P_{y,w}$ of [14]) satisfying $\Pi_{A, A} = 1$. The reader is referred to [17, §4] for full details, or to [11, §2.4] for a more elementary construction.

An element $A \in \mathfrak{S}_{r,n,n}$ is said to be *aperiodic* if for any $p \in \mathbb{Z} \setminus \{0\}$ there exists $k \in \mathbb{Z}$ such that $a_{k, k+p} = 0$. Let $\mathfrak{S}_{r,n,n}^{\text{ap}}$ be the set of aperiodic elements in $\mathfrak{S}_{r,n,n}$.

Theorem 1.5.2 (Lusztig) *Under the identifications of Theorem 1.4.7, the subalgebra $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ of $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ projects to the $\mathbb{Q}(v)$ -span of the elements*

$$\{\{A\} : A \in \mathfrak{S}_{r,n,n}^{\text{ap}}\}.$$

Proof This is [17, Theorem 8.2]. □

Remark 1.5.3 Theorem 1.5.2 is not true if we replace the canonical basis by one of the other two bases so far discussed.

If we have $n > r$, elementary considerations show that every element of $\mathfrak{S}_{r,n,n}$ is aperiodic. This means that the subalgebra of $\widehat{\mathbf{S}}_v(n, r)$ described in Theorem 1.5.2 is in fact the whole of $\widehat{\mathbf{S}}_v(n, r)$, so that we may refer to the algebra of Theorem 1.5.2 as “the affine q -Schur algebra” without confusion. We will concentrate on the case $n > r$ in this paper.

1.6 Main results

Our main aim is to prove the following

Theorem 1.6.1 *Let $n > r$, and identify $\widehat{\mathbf{S}}_v(n, r)$ with the quotient of $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ described in Theorem 1.5.2 (see Remark 1.5.3). Over $\mathbb{Q}(v)$, the affine v -Schur algebra $\widehat{\mathbf{S}}_v(n, r)$ is given by generators E_i, F_i, K_i, K_i^{-1} ($1 \leq i \leq n$) subject to relations (Q1) to (Q9) of Definition 1.4.2 (reading indices modulo n), together with the relations*

$$K_1 K_2 \cdots K_n = v^r \tag{Q15}$$

$$(K_i - 1)(K_i - v)(K_i - v^2) \cdots (K_i - v^r) = 0. \tag{Q16}$$

The corresponding result in finite type A was proved by the first author and A. Giaquinto. We will appeal to it repeatedly in the sequel.

Theorem 1.6.2 *Identify $\mathbf{S}_v(n, r)$ with the quotient of $\mathbf{U}(\mathfrak{gl}_n)$ described in Theorem 1.4.8. Over $\mathbb{Q}(v)$, the v -Schur algebra $\mathbf{S}_v(n, r)$ is given by generators E_i, F_i ($1 \leq i \leq n-1$) and K_i, K_i^{-1} ($1 \leq i \leq n$) subject to relations (Q1) to (Q9) of Definition 1.4.1, together with the relations*

$$K_1 K_2 \cdots K_n = v^r$$

$$(K_i - 1)(K_i - v)(K_i - v^2) \cdots (K_i - v^r) = 0.$$

Proof This is [7, Theorem 2.1]. □

Definition 1.6.3 For now, we will denote by T the $\mathbb{Q}(v)$ -algebra given by the generators and relations of Theorem 1.6.1, and we will denote the corresponding epimorphism from $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ to T by β . The main aim is thus to show that T is isomorphic to $\widehat{\mathbf{S}}_v(n, r)$.

Remark 1.6.4 There is an obvious isomorphism between the algebra given by the generators and relations of Theorem 1.6.2 and the subalgebra of T generated by the images of the the E_i, F_i, K_j and K_j^{-1} , where $1 \leq i < n$ and $1 \leq j \leq n$. This means that if a relation in $\widehat{\mathbf{S}}_v(n, r)$ involving the E_i, F_i and K_j avoids all occurrences of E_a and F_a for some $1 \leq a \leq n$, then by Theorem 1.6.2 and symmetry, the relation is a consequence of relations (Q15) and (Q16).

The following result establishes a natural surjection from T to $\widehat{\mathbf{S}}_v(n, r)$, and our main task in proving Theorem 1.6.1 will be to show that this map is an isomorphism, in other words, that relations (Q15) and (Q16) are sufficient.

Proposition 1.6.5 *Relations (Q15) and (Q16) of Theorem 1.6.1 hold in $\widehat{\mathbf{S}}_v(n, r)$, and therefore $\widehat{\mathbf{S}}_v(n, r)$ is a quotient of the algebra T . (We denote the corresponding epimorphism by $\gamma: T \rightarrow \widehat{\mathbf{S}}_v(n, r)$.)*

Proof Using the comultiplication on $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$, it may be easily checked that

$$K_1 K_2 \cdots K_n - v^r$$

and

$$(K_i - 1)(K_i - v)(K_i - v^2) \cdots (K_i - v^r)$$

act as zero on the tensor space $(V')^{\otimes r}$ given in Theorem 1.4.8. The result now follows from Lemma 1.4.11. \square

Remark 1.6.6 For later reference, we note that the maps α, β, γ respectively from Theorem 1.4.8, Remark 1.6.4, and Proposition 1.6.5 fit together into the following commutative diagram

$$\begin{array}{ccccc}
 & & \widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n) & & \\
 & & \uparrow & \searrow & \\
 & & \widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n) & \xrightarrow{\beta} & T & \xrightarrow{\gamma} & \widehat{\mathbf{S}}_v(n, r) \\
 & & \uparrow & & \uparrow & & \\
 & & \mathbf{U}(\widehat{\mathfrak{gl}}_n) & \xrightarrow{\alpha} & \mathbf{S}_v(n, r) & &
 \end{array}$$

in which all horizontal maps and the diagonal one are surjections, and all vertical maps are injections.

2 Proof of the main results

Most of this section is devoted to proving Theorem 1.6.1. The final result of this section, Theorem 2.6.1, is an equivalent formulation of Theorem 1.6.1, compatible with Lusztig's modified form of the quantized enveloping algebra.

2.1 A subalgebra of $\widehat{\mathbf{S}}_v(n, r)$ isomorphic to $\mathcal{H}(W)$

A presentation for $\widehat{\mathbf{S}}_v(n, r)$ in the case $n > r$ was given in [11, Proposition 2.5.1], and our main strategy for proving Theorem 1.6.1 will be to adapt this presentation.

Proposition 2.1.1 *The algebra $\widehat{\mathbf{S}}_v(n, r)$ is generated by elements*

$$\{\phi_{\omega, \omega}^d : d \in \widehat{W}\} \cup \{\phi_{\lambda, \omega}^1 : \lambda \in \Lambda(n, r)\} \cup \{\phi_{\omega, \lambda}^1 : \lambda \in \Lambda(n, r)\}.$$

The elements $\phi_{\omega, \omega}^d$ are subject to the relations of the affine Hecke algebra of Definition 1.2.1 under the identification given by Theorem 1.3.6 (iii). The generators are also subject to the following defining relations, where s denotes a generator $s_i \in \widehat{W}_\lambda$.

$$\phi_{\omega, \lambda}^1 \phi_{\mu, \omega}^1 = \delta_{\lambda, \mu} \sum_{d \in \widehat{W}_\lambda} \phi_{\omega, \omega}^d, \quad (\text{Q17})$$

$$\phi_{\omega, \omega}^s \phi_{\omega, \lambda}^1 = q \phi_{\omega, \lambda}^1, \quad (\text{Q18})$$

$$\phi_{\lambda, \omega}^1 \phi_{\omega, \omega}^s = q \phi_{\lambda, \omega}^1. \quad (\text{Q19})$$

A key step in understanding the structure of the algebra T of Definition 1.6.3 is locating within it a subalgebra isomorphic to the affine Hecke algebra $\mathcal{H}(\widehat{W})$. Theorem 1.3.6 (iii) shows that this can be done for the algebra $\widehat{\mathbf{S}}_v(n, r)$, and we now review how this works in terms of endomorphisms of tensor space. Recall the definition of weight space from Definition 1.4.6, and the definition of the weight ω from Definition 1.3.1.

Definition 2.1.2 For each $1 \leq i < r$, let $\tau(T_{s_i}) : V_\omega \rightarrow V_\omega$ be the endomorphism corresponding to the action of $vF_iE_i - 1 \in \widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$. Similarly, let $\tau(T_{\rho^{-1}})$ be the endomorphism corresponding to

$$F_n F_{n-1} \cdots F_{r+1} R,$$

and let $\tau(T_\rho)$ be the endomorphism corresponding to

$$E_r E_{r+1} \cdots E_{n-1} R^{-1}.$$

Lemma 2.1.3 *The endomorphisms $\tau(T_w)$ defined above (for $w \in \{s_i : 1 \leq i < r\} \cup \{\rho, \rho^{-1}\}$) satisfy the relations of Lemma 1.2.2 (after replacing T_w by $\tau(T_w)$).*

Proof Using the epimorphism $\alpha' : \mathbf{U}(\mathfrak{gl}_n) \twoheadrightarrow \mathbf{S}_v(n, r)$ studied in [1], [8], [10], one finds that the action of $\tau(T_{s_i})$ on V_ω in the case where $i \neq r$ corresponds to the action of $\phi_{\omega, \omega}^{s_i} \in \mathbf{S}_v(n, r)$. (Recall from Theorem 1.4.8 that $\mathbf{S}_v(n, r)$ is the quotient of $\mathbf{U}(\mathfrak{gl}_n)$ by the annihilator of $V_n^{\otimes r}$.) This proves relations (1'), (2') and (3') of Lemma 1.2.2.

The effect of $\tau(T_\rho)$ on V_ω is

$$\tau(T_\rho)(e_{i_1} \otimes \cdots \otimes e_{i_r}) = e_{j_1} \otimes \cdots \otimes e_{j_r},$$

where $j_t = i_t - 1 \pmod r$. The effect of $\tau(T_{\rho^{-1}})$ on V_ω is the inverse of this action. The proof of relations (4') and (5') now follow by calculation of the action of $vF_i E_i - 1$ on V_ω using the comultiplication. \square

Remark 2.1.4 Definition 2.1.2 and Lemma 2.1.3 are very similar to [11, Definition 3.3.1] and [11, Lemma 3.3.2], respectively. They are included here because [11, Definition 3.3.1] contains an incorrect definition for $\tau(T_{s_r})$.

Lemma 2.1.5 Define $\tau(T_{s_r}) := \tau(T_\rho)\tau(T_{s_1})\tau(T_{\rho^{-1}})$. Then the map taking $\tau(T_w)$ to T_w (where $w \in \{s_i : 1 \leq i \leq r\} \cup \{\rho, \rho^{-1}\}$) extends uniquely to an isomorphism of algebras between $\mathcal{H}(\widehat{W})$ and the algebra $\tau(\mathcal{H})$ generated by the endomorphisms $\tau(T_w)$.

Proof This follows from Lemma 1.2.2 and the argument given in [11], namely [11, Lemma 3.3.3, Lemma 3.3.4]. \square

For later purposes, it will be convenient to have versions of the above results that do not make reference to the grouplike elements R and R^{-1} . The following lemma is the key to the necessary modifications. (Recall that $n \geq r + 1$ by assumption.)

Lemma 2.1.6 Let $e \in V_\omega$. Then we have

$$R.e = (F_1 F_2 \cdots F_r).e$$

and

$$R^{-1}.e = (E_{r-1} E_{r-2} \cdots E_1) E_n.e.$$

Proof It is enough to consider the case where

$$e = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r}$$

is a basis element, and this turns out to be a straightforward exercise using the comultiplication in $\widehat{\mathbf{U}}(\mathfrak{gl}_n)$. \square

Proposition 2.1.7 For each $1 \leq i < r$, let $\tau'(T_{s_i}) : V_\omega \rightarrow V_\omega$ be the endomorphism corresponding to the action of $vF_i E_i - 1 \in \mathbf{U}(\widehat{\mathfrak{gl}_n})$. Similarly, let $\tau'(T_{\rho^{-1}})$ be the endomorphism corresponding to

$$(F_n F_{n-1} \cdots F_{r+1})(F_1 F_2 \cdots F_r),$$

and let $\tau'(T_p)$ be the endomorphism corresponding to

$$(E_r E_{r+1} \cdots E_{n-1})(E_{r-1} E_{r-2} \cdots E_1) E_n.$$

Then, after replacing T_w by $\tau'(T_w)$, these endomorphisms satisfy the relations of Lemma 1.2.2.

Proof Combine Lemma 2.1.6 with Lemma 2.1.3. \square

2.2 Weight space decomposition of T

An important property of the algebra T is that it possesses a decomposition into left and right weight spaces, similar to that enjoyed by the ordinary and affine q -Schur algebras.

Definition 2.2.1 An element $t \in T$ is said to be of *left weight* $\lambda \in \Lambda(n, r)$ if for each i with $1 \leq i \leq n$ we have

$$\beta(K_i).t = v^{\lambda_i} t.$$

where β is the map defined in Definition 1.6.3. There is an analogous definition for elements of *right weight* λ . The left (respectively, right) λ -*weight space* of T is the $\mathbb{Q}(v)$ -submodule spanned by all elements of left (respectively, right) weight λ .

Definition 2.2.2 For each $\lambda \in \Lambda(n, r)$, define the idempotent element $1_\lambda \in T$ by the image of $1_\lambda \in \mathbf{S}_v(n, r)$ under the canonical inclusion map from Remark 1.6.4. Here the 1_λ are the idempotents which were defined in [7, (3.4)]. The sum of the 1_λ , as λ varies over $\Lambda(n, r)$, is 1. Moreover, $1_\lambda 1_\mu = 0$ for $\lambda \neq \mu$, *i.e.* the idempotents are pairwise orthogonal.

Proposition 2.2.3 *The algebra T is the direct sum of its left λ -weight spaces, and the nonzero λ -weight spaces are indexed by the elements of $\Lambda(n, r)$.*

Proof Thanks to the above orthogonal decomposition of the identity in T , there is a direct sum decomposition

$$T = \bigoplus_{\lambda \in \Lambda(n, r)} 1_\lambda T.$$

Moreover, in $\mathbf{S}_v(n, r)$ we have the identity

$$\alpha(K_i)1_\lambda = \lambda_i 1_\lambda \quad (i = 1, \dots, n)$$

from [7, Proposition 8.3(a)], where α is the quotient map $\mathbf{U}(\mathfrak{gl}_n) \rightarrow \mathbf{S}_v(n, r)$ of Theorem 1.4.8. Now it follows from the embedding of Remark 1.6.4, or more precisely from the commutativity of the diagram in Remark 1.6.6, that

$$\beta(K_i)1_\lambda = \lambda_i 1_\lambda \quad (i = 1, \dots, n)$$

holds in the algebra T . Thus it follows that $\beta(K_i)v = \lambda_i v$ for all $i = 1, \dots, n$ and all $v \in 1_\lambda T$. This proves that $1_\lambda T$ is the λ -weight space in T . \square

For simplicity's sake, we will write E_i in place of $\beta(E_i)$ and F_i in place of $\beta(F_i)$ for the remainder of §2.2.

- Lemma 2.2.4** (i) In T we have $K_i^{\pm 1} 1_\lambda = v^{\pm \lambda_i} 1_\lambda$.
(ii) The idempotent 1_λ lies within the subalgebra of T generated by the K_i .
(iii) In T , the idempotent 1_λ coincides with the image of $\phi_{\lambda, \lambda}^1$ under β .

Proof Part (i) is already contained in the proof of the preceding proposition, and part (ii) is due to the definition of 1_λ in [7] as

$$1_\lambda = \begin{bmatrix} K_1 \\ \lambda_1 \end{bmatrix} \cdots \begin{bmatrix} K_n \\ \lambda_n \end{bmatrix}.$$

where $\begin{bmatrix} K_i \\ \lambda_i \end{bmatrix} = \prod_{s=1}^t \frac{K_i v^{-s+1} - K_i^{-1} v^{s-1}}{v^s - v^{-s}}$.

Part (iii) is a consequence of the remarks preceding [10, Lemma 2.9] combined with [10, Corollary 2.10]. \square

Definition 2.2.5 For each i with $1 \leq i \leq n$, let $\alpha_i = ((\alpha_i)_1, \dots, (\alpha_i)_n)$ be the n -tuple of integers given by

$$(\alpha_i)_j = \begin{cases} 1 & \text{if } j \equiv i \pmod{n}, \\ -1 & \text{if } j \equiv i+1 \pmod{n}, \\ 0 & \text{otherwise.} \end{cases}$$

The following identities will be used frequently in the sequel, often without explicit reference. In these identities, it will be convenient to regard a weight $\lambda = (\lambda_1, \dots, \lambda_n)$ as an infinite periodic sequence of integers, indexed by \mathbb{Z} , by setting λ_j for any $j \in \mathbb{Z}$ to the corresponding value λ_i such that $1 \leq i \leq n$ and $j \equiv i \pmod{n}$.

Lemma 2.2.6 Let $\lambda \in \Lambda(n, r)$, extended to an infinite periodic sequence as above. The following identities hold in T :

(i) For any $1 \leq i \leq n$, we have

$$E_i 1_\lambda = \begin{cases} 1_{\lambda + \alpha_i} E_i & \text{if } \lambda_{i+1} > 0; \\ 0 & \text{otherwise.} \end{cases}$$

(ii) For any $1 \leq i \leq n$, we have

$$F_i 1_\lambda = \begin{cases} 1_{\lambda - \alpha_i} F_i & \text{if } \lambda_i > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Proof By Remark 1.6.4 and Lemma 2.2.4, it is enough to check that both sides of each identity agree after projection to $\widehat{\mathbf{S}}_v(n, r)$. By Theorem 1.4.7, it is enough to check that both sides of each identity agree in their action on tensor space, which is a routine calculation. \square

The following lemma will be used extensively in the sequel. We will sometimes refer to it as the *cancellation principle* for T .

Lemma 2.2.7 *Let $c \geq 1$. Let $\lambda \in \Lambda(n, r)$, extended to an infinite periodic sequence as above. The following identities hold in T :*

(i) *For each $1 \leq i \leq n$ with $\lambda_i = 0$, there exists a nonzero element $z \in \mathcal{A}$ such that*

$$F_i^c E_i^c 1_\lambda = \begin{cases} z 1_\lambda & \text{if } \lambda_{i+1} \geq c; \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, if $c = \lambda_{i+1} = 1$ then $z = 1$.

(ii) *For each $1 \leq i \leq n$ with $\lambda_{i+1} = 0$, there exists a nonzero $z' \in \mathcal{A}$ such that*

$$E_i^c F_i^c 1_\lambda = \begin{cases} z' 1_\lambda & \text{if } \lambda_i \geq c; \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, if $c = \lambda_i = 1$, then $z' = 1$.

Proof By the formulas in [16, 3.1.9] we have the following identities in $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$:

$$\begin{aligned} E_i^{(c)} F_i^{(c)} &= \sum_{t \geq 0} F_i^{(c-t)} \prod_{s=1}^t \frac{v^{2t-2c-s+1} \widetilde{K}_i - v^{-2t+2c+s-1} \widetilde{K}_i^{-1}}{v^s - v^{-s}} E_i^{(c-t)} \\ F_i^{(c)} E_i^{(c)} &= \sum_{t \geq 0} E_i^{(c-t)} \prod_{s=1}^t \frac{v^{2t-2c-s+1} \widetilde{K}_i^{-1} - v^{-2t+2c+s-1} \widetilde{K}_i}{v^s - v^{-s}} F_i^{(c-t)} \end{aligned}$$

where $\widetilde{K}_i = K_i K_{i+1}^{-1}$ and $X^{(m)} = X^m / [m]!$ for $X = E_i, F_i$. Here $[m]$ is the quantum integer $[m] = (v^m - v^{-m}) / (v - v^{-1})$ and $[m]! = [1] \cdots [m-1][m]$ for any $m \in \mathbb{N}$.

Since the above identities hold in $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$, they hold in the quotient T . Multiply the second identity on the right by 1_λ . By Lemma 2.2.6 and the hypothesis $\lambda_i = 0$ all terms on the right hand side will then vanish, excepting the term corresponding to $t = c$. So we obtain the identity

$$F_i^{(c)} E_i^{(c)} 1_\lambda = \prod_{s=1}^c \frac{v^{-s+1} \widetilde{K}_i^{-1} - v^{s-1} \widetilde{K}_i}{v^s - v^{-s}} 1_\lambda$$

and a similar argument with the first identity above in light of the hypothesis $\lambda_{i+1} = 0$ yields the identity

$$E_i^{(c)} F_i^{(c)} 1_\lambda = \prod_{s=1}^c \frac{v^{-s+1} \widetilde{K}_i - v^{s-1} \widetilde{K}_i^{-1}}{v^s - v^{-s}} 1_\lambda.$$

These are identities in the quotient T . In fact, they hold in the subalgebra $\mathbf{S}_v(n, r)$ under the embedding of Remark 1.6.4. By Lemma 2.2.4(i) the above identities in T take the form

$$\begin{aligned} F_i^{(c)} E_i^{(c)} 1_\lambda &= \prod_{s=1}^c \frac{v^{\lambda_{i+1} - \lambda_i - s + 1} - v^{\lambda_i - \lambda_{i+1} + s - 1}}{v^s - v^{-s}} 1_\lambda \\ E_i^{(c)} F_i^{(c)} 1_\lambda &= \prod_{s=1}^c \frac{v^{\lambda_i - \lambda_{i+1} - s + 1} - v^{\lambda_{i+1} - \lambda_i - s - 1}}{v^s - v^{-s}} 1_\lambda. \end{aligned}$$

Remembering that $\lambda_i = 0$ in the first formula and $\lambda_{i+1} = 0$ in the second, by multiplying through by $([c]!)^2$ we obtain the desired result, where

$$z = ([c]!)^2 \begin{bmatrix} \lambda_{i+1} \\ c \end{bmatrix}, \quad z' = ([c]!)^2 \begin{bmatrix} \lambda_i \\ c \end{bmatrix}.$$

in terms of the standard Gaussian binomial coefficients (see e.g. [16, §1.3]). The proof is complete. \square

Definition 2.2.8 Maintain the notation of Lemma 2.2.7. Let M be a monomial in the various elements E_i, F_i and 1_λ of T . We call a monomial M' a *reduction* of M if it (a) represents the same element of T as M and (b) M' can be obtained from M by omitting zero or more generators of M of the form 1_μ .

A *distinguished term* in the algebra T is an element of T of one of the following two forms:

- (i) $E_i^c 1_\lambda$, where $c \geq 0$ and $\lambda_i = 0$;
- (ii) $F_i^c 1_\lambda$, where $c \geq 0$ and $\lambda_{i+1} = 0$.

A *strictly distinguished monomial* in the algebra T is a monomial in the elements F_i, E_i and 1_λ that can be parsed as a word in the distinguished terms. A reduction of a strictly distinguished monomial is called a *distinguished monomial*.

Example 2.2.9 The idempotents 1_λ are both distinguished terms and distinguished monomials in T : here, we take $c = 0$.

If $\lambda_i = 0$ and $\lambda_{i+1} = c$ then the element $M' = F_i^c E_i^c 1_\lambda$ of Lemma 2.2.7 (i) is a distinguished monomial. Indeed, it can be seen by repeated applications of Lemma 2.2.6 (i) and the fact that 1_λ is idempotent, that M' is a reduction of the strictly distinguished monomial $M = (F_i^c 1_{\lambda+c\alpha_i})(E_i^c 1_\lambda)$. (To verify that M is strictly distinguished, one must note that $(\lambda + c\alpha_i)_{i+1} = 0$.) Furthermore, by Lemma 2.2.7 and the fact that $\lambda_{i+1} \geq c$, we see that $M = M'$ is a nonzero element of T .

Similarly, if $\lambda_i = c$ and $\lambda_{i+1} = 0$ then the element $E_i^c F_i^c 1_\lambda$ of Lemma 2.2.7 (ii) is a distinguished monomial.

In the sequel, we will make use of various automorphisms of T ; part (i) below may be used without explicit comment.

Proposition 2.2.10 (i) *There is a unique automorphism ν of T of order n satisfying*

$$\begin{aligned} \nu(E_i) &= E_{i+1}, \\ \nu(F_i) &= F_{i+1} \text{ and} \\ \nu(K_i^{\pm 1}) &= K_{i+1}^{\pm 1}, \end{aligned}$$

for all $1 \leq i \leq n$, and reading subscripts modulo n . Let $\lambda \in \Lambda(n, r)$ and define

$$\lambda_+ = (\lambda_n, \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{n-1}).$$

Then $\nu(1_\lambda) = 1_{\lambda_+}$.

(ii) *There is a unique anti-automorphism σ of T satisfying*

$$\sigma(E_i) = F_i,$$

$$\begin{aligned}\sigma(F_i) &= E_i \text{ and} \\ \sigma(K_i^{\pm 1}) &= K_i^{\pm 1},\end{aligned}$$

for all $1 \leq i \leq n$. The anti-automorphism σ fixes all elements $1_\lambda \in T$.

Proof For (i), we note that there is an automorphism of $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ corresponding to ν , that in addition fixes the elements $R^{\pm 1}$; this can be verified by checking the defining relations for $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$. Since this automorphism preserves setwise the set of relations (Q15) and (Q16) in T , we obtain an automorphism of T as claimed; it is unique because we have given its effect on a generating set (see Theorem 1.6.1). The last claim of (i) follows from the relationship between the K_i and 1_λ ; see for example [10, Corollary 2.10].

The same line of argument can be used to prove (ii). \square

Lemma 2.2.11 *Let $M = t_1 t_2 \cdots t_k$ be a strictly distinguished monomial with distinguished terms t_i . Then $M \neq 0$ if and only if the following two conditions hold:*

- (i) *each term t_i is nonzero;*
- (ii) *for each $1 \leq i < k$, there exists $\lambda \in \Lambda(n, r)$ such that $t_i = t_i 1_\lambda$ and $t_{i+1} = 1_\lambda t_{i+1}$.*

Proof Condition (i) is clearly necessary for M to be nonzero. To see the necessity of condition (ii), recall from Lemma 2.2.6 that for each term t_i , there exist $\lambda, \mu \in \Lambda(n, r)$ such that $t_i = 1_\lambda t_i = t_i 1_\mu$.

We now check sufficiency. It will be enough to show that $\sigma(M)M \neq 0$, where σ is as in Proposition 2.2.10. This follows from Lemma 2.2.7. Indeed, the hypotheses $\lambda_{i+1} \geq c$ or $\lambda_i \geq c$ follow from condition (i) above, and condition (ii) above implies that if $t_i 1_\lambda = t_i$ then we have

$$\begin{aligned}\sigma(t_{i+1})\sigma(t_i)t_i t_{i+1} &= z'' \sigma(t_{i+1})1_\lambda t_{i+1} \\ &= z'' \sigma(t_{i+1})t_{i+1},\end{aligned}$$

where z'' is equal either to z or to z' as in Lemma 2.2.7. There is a unique $\mu \in \Lambda(n, r)$ such that $M = M 1_\mu$, and an induction now shows that $\sigma(M)M$ is a nonzero scalar multiple of 1_μ , completing the proof. \square

2.3 A subalgebra of T isomorphic to $\widehat{\mathbf{S}}_\nu(n, r)$

The aim of §2.3 is to show that the relations satisfied by the endomorphisms of Proposition 2.1.7 are in fact consequences of the defining relations (Q15) and (Q16) of the algebra T . In this section, we may abuse notation by identifying elements u of $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ with their images $\beta(u)$ in T (see Definition 1.6.3).

Recall from Remark 1.6.4 that there is a natural subalgebra of T isomorphic to the ordinary ν -Schur algebra, $\mathbf{S}_\nu(n, r)$. Using this fact, we can make the following

Definition 2.3.1 For each $1 \leq i < r$, define elements of T by

$$\begin{aligned}\zeta(T_{s_i}) &= (\nu F_i E_i - 1)1_\omega, \\ \zeta(T_{\rho^{-1}}) &= ((F_n F_{n-1} \cdots F_{r+1})(F_1 F_2 \cdots F_r))1_\omega, \\ \zeta(T_\rho) &= ((E_r E_{r+1} \cdots E_{n-1})(E_{r-1} E_{r-2} \cdots E_1)E_n)1_\omega.\end{aligned}$$

Remark 2.3.2 It follows from repeated applications of Lemma 2.2.6 that each element $\zeta(T_w)$ given in Definition 2.3.1 has the property that $1_\omega \zeta(T_w) = \zeta(T_w)$.

Lemma 2.3.3 *The expressions given for $\zeta(T_{\rho-1})$ and $\zeta(T_\rho)$ are distinguished monomials.*

Proof This is a routine exercise, in which the hypothesis that $n > r$ plays an important part. \square

Lemma 2.3.4 *The following identities hold in T :*

- (i) $\zeta(T_{\rho-1}) = (F_n(F_1F_2 \cdots F_{r-2}F_{r-1})(F_{n-1}F_{n-2} \cdots F_{r+1}F_r))1_\omega$;
- (ii) $\zeta(T_\rho) = ((E_rE_{r-1} \cdots E_2E_1)(E_{r+1}E_{r+2} \cdots E_{n-1}E_n))1_\omega$.

Proof Equation (i) (respectively, (ii)) follows by applying repeated commutations between the generators F_i (respectively, E_i). \square

Lemma 2.3.5 *The following identities hold in T :*

- (i) $\zeta(T_{\rho-1})\zeta(T_\rho) = 1_\omega$;
- (ii) $\zeta(T_\rho)\zeta(T_{\rho-1}) = 1_\omega$.

Proof Let $\omega' \in \Lambda(n, r)$ be the weight $\omega' = (0, 1, \dots, 1, 0, \dots, 0)$, where the occurrences of 1 appear in positions $2, 3, 4, \dots, r+1 \leq n$. Then it follows from Lemma 2.3.4 that

$$\zeta(T_\rho) = 1_\omega ((E_rE_{r-1} \cdots E_2E_1)1_{\omega'}(E_{r+1}E_{r+2} \cdots E_{n-1}E_n))1_\omega$$

and it follows from Definition 2.3.1 that

$$\zeta(T_{\rho-1}) = 1_\omega ((F_nF_{n-1} \cdots F_{r+1})1_{\omega'}(F_1F_2 \cdots F_r))1_\omega.$$

We will prove (i), and (ii) follows by a similar argument.

To prove (i), we first show that

$$1_{\omega'}(F_1F_2 \cdots F_r)1_\omega(E_rE_{r-1} \cdots E_2E_1)1_{\omega'} = 1_{\omega'}.$$

The left hand side of the equation is readily checked to be a good monomial, and then the equation follows by repeated applications of the $c = 1$ case of the cancellation principle (Lemma 2.2.7), starting in the middle of the equation (i.e., with $F_r1_\omega E_r$). A similar argument shows that

$$1_\omega(F_nF_{n-1} \cdots F_{r+1})1_{\omega'}(E_{r+1}E_{r+2} \cdots E_{n-1}E_n)1_\omega = 1_\omega.$$

Part (i) follows by combining these last two identities. \square

Lemma 2.3.6 *Let $1 < i < r$, and let*

$$M = (E_{r-1}E_{r-2} \cdots E_1)(E_{r+1}E_{r+2} \cdots E_n)1_\omega.$$

Then the identity $(vF_{i-1}E_{i-1} - 1)M = M(vF_iE_i - 1)$ holds in T .

Note. Notice that both sides of the identity have right weight ω .

Proof Since the identity involves no occurrences of E_r or F_r , Remark 1.6.4 applies. More precisely, after applying a suitable symmetry of the Dynkin diagram, we see that it suffices to prove the identity

$$(vF_{n-r+i-1}E_{n-r+i-1} - 1)M' = M'(vF_{n-r+i}E_{n-r+i} - 1)$$

in the ordinary v -Schur algebra, where indices are read modulo n , and we have

$$M' = (E_{n-1}E_{n-2} \cdots E_{n-r+1})(E_1E_2 \cdots E_{n-r})1_\omega,$$

and

$$\omega' = (0, 0, \dots, 0, 1, 1, \dots, 1),$$

where 1 occurs r times in ω' .

By Theorem 1.4.8, it suffices to show that both sides of the identity act in the same way on tensor space $V'^{\otimes r}$, and because both sides of the identity have right weight ω' , it is enough to check this on the ω' -weight space. By Lemma 1.4.10, it is enough to check that each side of the identity acts the same on the element

$$e_{\omega'} = e_{n-r+1} \otimes e_{n-r+2} \otimes \cdots \otimes e_n.$$

Fix j with $1 \leq j < r$, and let $e_{j,\omega'}$ be the tensor obtained by exchanging the occurrences of e_{n-r+j} and $e_{n-r+j+1}$ in $e_{\omega'}$. Using the comultiplication, it is a routine calculation to show that

$$(F_{n-r+j}E_{n-r+j})e_{\omega} = e_{j,\omega'} + v^{-1}e_{\omega'},$$

and it is immediate from this that

$$(vF_{n-r+j}E_{n-r+j} - 1)e_{\omega} = ve_{j,\omega'}.$$

Another calculation with the comultiplication shows that

$$(E_{n-1}E_{n-2} \cdots E_{n-r+1})(E_1E_2 \cdots E_{n-r})1_{\omega'}e_{\omega'} = e'_{\omega'},$$

where

$$e'_{\omega'} = e_1 \otimes (e_{n-r+1} \otimes e_{n-r+2} \otimes \cdots \otimes e_{n-1}).$$

Let j be such that $1 \leq j < r$. Letting $(vF_{n-r+j}E_{n-r+j} - 1)$ act on the left, we deduce that

$$(vF_{n-r+j}E_{n-r+j} - 1)(E_{n-1}E_{n-2} \cdots E_{n-r+1})(E_1E_2 \cdots E_{n-r})1_{\omega'}e'_{\omega'} = ve'_{j,\omega'},$$

where $e'_{j,\omega'}$ is obtained from $e_{j,\omega}$ by exchanging the occurrences of e_{n-r+j} and $e_{n-r+j+1}$.

The result now follows after we observe that

$$(E_{n-1}E_{n-2} \cdots E_{n-r+1})(E_1E_2 \cdots E_{n-r})1_{\omega'}e_{j+1,\omega'} = e'_{j,\omega'}.$$

□

Corollary 2.3.7 *If i is such that $1 < i < r$, then the relation*

$$\zeta(T_{s_{i-1}})\zeta(T_\rho) = \zeta(T_\rho)\zeta(T_{s_i})$$

holds in T .

Proof Observe that E_r commutes with F_i , F_{i-1} , E_i and E_{i-1} . The assertion now follows by left-multiplying the identity of Lemma 2.3.6 by E_r . \square

The techniques of proof of Lemma 2.3.6 play an important part in the next brace of results.

Lemma 2.3.8 *The following identities hold in T , where $1 < i < r$:*

- (i) $(F_i E_i - v^{-1})1_\omega = (E_i F_i - v)1_\omega$;
- (ii) $1_\omega(F_n F_{n-1} \cdots F_r E_r E_{r+1} \cdots E_n - v^{-1}) = 1_\omega(E_r E_{r+1} \cdots E_n F_n F_{n-1} \cdots F_r - v)$;
- (iii) $(E_n F_n - v)E_1 E_n 1_\omega = E_1 E_n (E_1 F_1 - v)1_\omega$;
- (iv) $1_\omega(F_{r-1} E_{r-1} - v^{-1})E_r E_{r-1} \cdots E_2 E_1 = 1_\omega E_r E_{r-1} \cdots E_2 E_1 (F_r E_r - v^{-1})$.

Note. The expressions appearing in (i) and (ii) above have both left and right weight equal to ω .

Proof We omit the proof of (i), because it is similar to, but easier than, the proof of (ii).

To prove (ii), it is enough, by symmetry of the defining relations of T , to prove the identity

$$1_{\omega^-}(F_{n-1} \cdots F_{r-1} E_{r-1} \cdots E_{n-1} - v^{-1}) = 1_{\omega^-}(E_{r-1} \cdots E_{n-1} F_{n-1} \cdots F_{r-1} - v),$$

where

$$\omega^- = (\underbrace{1, 1, \dots, 1}_{r-1}, \underbrace{0, 0, \dots, 0}_{n-r}, 1).$$

This can be regarded as an identity in $\mathbf{S}_v(n, r)$. By Lemma 1.4.10, it is enough to check that each side of the identity acts in the same way on the element

$$e_{\omega^-} = e_1 \otimes e_2 \otimes \cdots \otimes e_{r-2} \otimes e_{r-1} \otimes e_n.$$

A calculation shows that each side of the identity acts on e_{ω^-} to give

$$e_1 \otimes e_2 \otimes \cdots \otimes e_{r-2} \otimes e_n \otimes e_{r-1}.$$

To prove (iii), it is enough by symmetry of the defining relations to prove

$$(E_1 F_1 - v)E_2 E_1 1_{\omega^+} = E_2 E_1 (E_2 F_2 - v)1_{\omega^+},$$

where

$$\omega^+ = (0, \underbrace{1, 1, \dots, 1}_r, \underbrace{0, 0, \dots, 0}_{n-r-1}).$$

By Lemma 1.4.10, it is enough to show that both sides of the identity act in the same way on

$$e_{\omega^+} = e_2 \otimes e_3 \otimes \cdots \otimes e_{r+1}.$$

A calculation shows that each side sends e_{ω^+} to

$$e_2 \otimes e_1 \otimes e_4 \otimes e_5 \otimes \cdots \otimes e_{r+1}.$$

For (iv), observe that both sides of the identity have right weight ω^+ , as defined above. Since (iv) can be regarded as an identity in $\mathbf{S}_v(n, r)$, Lemma 1.4.10 applies

and it is enough to check that both sides of the identity have the same effect on e_{ω^+} . A calculation shows that both sides of the identity send e_{ω^+} to

$$e_1 \otimes e_2 \otimes \cdots \otimes e_{r-3} \otimes e_{r-2} \otimes e_r \otimes e_{r-1}.$$

□

Definition 2.3.9 We define $\zeta(T_{s_r})$ to be the element of T given by

$$1_{\omega}(F_n F_{n-1} \cdots F_r E_r E_{r+1} \cdots E_n - v^{-1}).$$

Lemma 2.3.10 *The following identities hold in T :*

- (i) $\zeta(T_{\rho})\zeta(T_{s_r}) = \zeta(T_{s_{r-1}})\zeta(T_{\rho})$;
- (ii) $\zeta(T_{\rho})\zeta(T_{s_1}) = \zeta(T_{s_r})\zeta(T_{\rho})$.

Proof We prove (i) first. Using Lemma 2.3.4(ii), it is enough to prove that the expressions

$$M_1 = 1_{\omega}(E_r E_{r-1} \cdots E_1)(E_{r+1} E_{r+2} \cdots E_n)(F_n F_{n-1} \cdots F_{r+1} F_r)(E_r E_{r+1} \cdots E_n)$$

and

$$M_2 = 1_{\omega}(F_{r-1} E_{r-1})(E_r E_{r-1} \cdots E_1)(E_{r+1} E_{r+2} \cdots E_n)$$

are equal.

Using Lemma 2.2.6 repeatedly, and the notation of the proof of Lemma 2.3.6, we find that

$$M_1 = 1_{\omega} E_r E_{r-1} \cdots E_1 (1_{\omega^+} E_{r+1} E_{r+2} \cdots E_n F_n F_{n-1} \cdots F_{r+1} 1_{\omega^+}) F_r E_r E_{r+1} \cdots E_n,$$

and repeated applications of the cancellation principle (Lemma 2.2.7) show that the given parenthetical expression is identically equal to 1_{ω^+} . By Lemma 2.2.6 again, we have

$$\begin{aligned} M_1 &= 1_{\omega} E_r E_{r-1} \cdots E_1 1_{\omega^+} F_r E_r E_{r+1} \cdots E_n \\ &= 1_{\omega}(E_r E_{r-1} \cdots E_1) F_r E_r (E_{r+1} E_{r+2} \cdots E_n). \end{aligned}$$

Applying Lemma 2.3.8 (iv) gives

$$M_1 = 1_{\omega} F_{r-1} E_{r-1} (E_r E_{r-1} \cdots E_1) (E_{r+1} E_{r+2} \cdots E_n),$$

which is M_2 , as required.

We now turn to (ii). By Lemma 2.3.8, parts (i) and (ii), it is enough to show that the monomials

$$M_3 = 1_{\omega}(E_r E_{r+1} \cdots E_n)(F_n F_{n-1} \cdots F_r)(E_r E_{r+1} \cdots E_{n-1})(E_{r-1} E_{r-2} \cdots E_1) E_n$$

and

$$M_4 = 1_{\omega}(E_r E_{r+1} \cdots E_{n-1})(E_{r-1} E_{r-2} \cdots E_1) E_n (E_1 F_1)$$

are equal. Using Lemma 2.2.6 and the notation of the proof of Lemma 2.3.6 again, we find that M_3 is equal to

$$1_{\omega} E_r E_{r+1} \cdots E_n F_n (1_{\omega^-} F_{n-1} F_{n-2} \cdots F_r E_r E_{r+1} \cdots E_{n-1} 1_{\omega^-}) E_{r-1} E_{r-2} \cdots E_1 E_n.$$

By the cancellation principle, this simplifies to

$$\begin{aligned} M_3 &= 1_\omega E_r E_{r+1} \cdots E_n F_n 1_{\omega^{-1} E_{r-1} E_{r-2} \cdots E_1 E_n} \\ &= 1_\omega (E_r E_{r+1} \cdots E_{n-1}) (E_n F_n) (E_{r-1} E_{r-2} \cdots E_1) E_n. \end{aligned}$$

Applying commutations yields

$$M_3 = 1_\omega (E_r E_{r+1} \cdots E_{n-1}) (E_{r-1} E_{r-2} \cdots E_2) (E_n F_n E_1 E_n).$$

Using Lemma 2.3.8 (iii), we have

$$M_3 = 1_\omega (E_r E_{r+1} \cdots E_{n-1}) (E_{r-1} E_{r-2} \cdots E_2) (E_1 E_n E_1 F_1),$$

which is equal to M_4 , as desired. \square

Proposition 2.3.11 *Let $\gamma: T \rightarrow \widehat{\mathbf{S}}_v(n, r)$ be the epimorphism of Proposition 1.6.5. Then γ admits a right inverse: there is an injective homomorphism*

$$\iota: \widehat{\mathbf{S}}_v(n, r) \rightarrow T$$

such that $\gamma \circ \iota$ is the identity homomorphism on $\widehat{\mathbf{S}}_v(n, r)$.

Proof We start by specifying ι on the subalgebra of $\widehat{\mathbf{S}}_v(n, r)$ spanned by the elements $\phi_{\omega, \omega}^d$, as in §2.1. We define

$$\iota(\tau(T_{s_i})) := \zeta(T_{s_i})$$

for $1 \leq i < r$, and define

$$\iota(\tau(T_{\rho^{\pm 1}})) := \zeta(T_{\rho^{\pm 1}}).$$

In these cases, $\gamma \circ \iota$ is the identity map by Proposition 2.1.7 and the definition of ζ , so it is enough to check that the relations of Lemma 1.2.2 are satisfied in the image of ι . The difficult cases, (4') and (5'), follow from Lemma 2.3.5, Corollary 2.3.7 and Lemma 2.3.10. Since those cases hold, it is enough to check cases (1'), (2') and (3') assuming that neither s nor t is equal to r ; this follows from Lemma 2.1.3 and Remark 1.6.4.

It remains to check relations (Q17), (Q18) and (Q19) of Proposition 2.1.1. Since, by Remark 1.6.4, there is a canonically embedded copy of $\mathbf{S}_v(n, r)$ in the algebra T (namely, the subalgebra generated by all $E_i, F_i, K_j^{\pm 1}$ with $i \neq n$), we may send the elements $\phi_{\omega, \lambda}^1, \phi_{\mu, \omega}^1$ and $\phi_{\omega, \omega}^d$ (where d lies in the finite symmetric group) to the corresponding elements of T . (Observe that this construction is compatible with the definitions in the previous paragraph.) More explicitly, if M is a polynomial in the generators E_i, F_i, K_j, K_j^{-1} and there are no occurrences of E_n or F_n , then $\alpha(M) \in \mathbf{S}_v(n, r)$ and $\iota(\alpha(M)) = \beta(M)$ by construction. It then follows that

$$(\gamma \circ \iota)(\alpha(M)) = \gamma(\beta(M)) = \alpha(M),$$

as required. Relations (Q17), (Q18) and (Q19) can be seen to hold in T (and thus in the image of ι) by Theorem 1.6.2 and Remark 1.6.4. \square

2.4 Surjectivity of ι

So far we have shown that there is a monomorphism $\iota : \widehat{\mathbf{S}}_v(n, r) \longrightarrow T$. From Definition 1.6.3 we have a surjective map $\beta : \mathbf{U}(\widehat{\mathfrak{gl}}_n) \longrightarrow T$. We aim in §2.4 to show that the image of β is contained in the image of ι , which will complete the proof of Theorem 1.6.1.

Lemma 2.4.1 *The algebra T is generated by the all elements of the form $E_i 1_\lambda$, $1_\lambda F_i$ and 1_λ , for $1 \leq i \leq n$ and $\lambda \in \Lambda(n, r)$.*

Proof The images of the K_i and K_i^{-1} in T are linear combinations of the elements 1_λ , as they are in the ordinary v -Schur algebra (see [10, Corollary 2.10]). The image of E_i in T is a linear combination of elements $E_i 1_\lambda$, because the 1_λ form an orthogonal decomposition of the identity (see Lemma 2.2.4). Similarly, the image of F_i in T is a linear combination of elements $1_\lambda F_i$. This shows that the usual algebra generators of T lie in the span of the elements listed in the statement. Conversely, it follows from the definitions (see Definition 2.2.2) that the elements listed lie in T , completing the proof. \square

Lemma 2.4.2 *If $i \neq n$ and $\lambda \in \Lambda(n, r)$, the elements $E_i 1_\lambda$, $1_\lambda F_i$ and 1_λ lie in the image of ι .*

Proof The elements listed in the statement lie in the canonically embedded copy of $\mathbf{S}_v(n, r)$ in T . By the construction of ι (see the proof of Proposition 2.3.11), such elements lie in the image of ι . \square

Lemma 2.4.3 (i) *The element $E_n 1_\omega$ of T lies in $\iota(\widehat{\mathbf{S}}_v(n, r))$.*

(ii) *The element $1_\omega F_n$ of T lies in $\iota(\widehat{\mathbf{S}}_v(n, r))$.*

Proof By construction of ι , the element $\zeta(T_\rho)$ lies in the image of ι . Since T contains a canonical copy of $\mathbf{S}_v(n, r)$ (see the proof of Proposition 2.3.11), the element

$$(F_1 F_2 \cdots F_{r-1})(F_{n-1} F_{n-2} \cdots F_r) 1_\omega$$

lies in the image of ι . By multiplying these two elements we see that

$$(F_1 F_2 \cdots F_{r-1})(F_{n-1} F_{n-2} \cdots F_r)(E_r E_{r+1} \cdots E_{n-1})(E_{r-1} E_{r-2} \cdots E_1) E_n 1_\omega$$

lies in the image of ι . Applying the cancellation principle, this latter expression simplifies to $E_n 1_\omega$, completing the proof of (i).

A similar argument using $\zeta(T_{\rho^{-1}})$ in place of $\zeta(T_\rho)$ can be used to prove (ii). \square

Our main effort will be directed towards proving that the elements $E_n 1_\lambda$ lie in the image of ι . More precisely, we will prove that $E_n 1_\lambda$ lies in the ideal of $\text{Im}(\iota)$ generated by $E_n 1_\omega$. Our argument will rely on the following technical lemma whose proof will be deferred to §2.5.

Lemma 2.4.4 Fix $\lambda \in \Lambda(n, r)$ such that $\lambda_1 > 0$. There exists a distinguished monomial M in the generators E_i, F_i and 1_μ satisfying the following conditions:

- (i) $M = 1_\omega M 1_\lambda \neq 0$;
- (ii) M contains no occurrences of E_n, F_n, E_1 or F_{n-1} ;
- (iii) all the occurrences of F_1 occur consecutively, as do all the occurrences of E_{n-1} ;
- (iv) there are at most $\lambda_1 - 1$ occurrences of F_1 .

Lemma 2.4.5 Let σ be the antiautomorphism of T given in Proposition 2.2.10 (ii), and let M and λ be as in Lemma 2.4.4. Then there is a nonzero scalar $z \in \mathbb{Q}(v)$ such that

$$E_n 1_\lambda = z \sigma(M) (E_n 1_\omega) M.$$

In particular, $E_n 1_\lambda$ lies in the ideal of $\text{Im}(\iota)$ generated by $E_n 1_\omega$.

Proof The monomial M is equal to a strictly distinguished monomial

$$M' = t'_m t'_{m-1} \cdots t'_1.$$

After moving unnecessary idempotents in M to the right using Lemma 2.2.7, and omitting the corresponding idempotents from the terms t'_i , we may assume that M is of the form

$$M = t_m t_{m-1} \cdots t_1 1_\lambda.$$

We will prove by induction on $k \leq m$ that

$$(\sigma(t_1) \sigma(t_2) \cdots \sigma(t_k)) E_n (t_k t_{k-1} \cdots t_1) 1_\lambda$$

is a nonzero multiple of $E_n 1_\lambda$; the case $k = m$ is the assertion of the Lemma. The base case, $k = 0$, is trivial.

There are two cases to consider for the inductive step. The first case, which is easier to deal with, is that t_k is of the form E_i^c for some $c > 0$. In this case, we have

$$\sigma(t_k) E_n t_k = F_i^c E_n E_i^c,$$

which can be rewritten as

$$E_n F_i^c E_i^c$$

using the relations of $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$. (Note that we do not have $i = n$, because M does not contain occurrences of E_n .) We now have

$$F_i^c E_n t_k t_{k-1} \cdots t_1 1_\lambda = E_n F_i^c t_k 1_\mu t_{k-1} t_{k-2} \cdots t_1 1_\lambda$$

for a suitable $\mu \in \Lambda(n, r)$. The hypothesis $M \neq 0$ means that we have $\mu_{i+1} \geq c$, so we may apply Lemma 2.2.7 (i) to replace $F_i^c t_k 1_\mu$ by $z 1_\mu$ with z nonzero. The proof is now completed in this case by the inductive hypothesis.

The second case is that t_k is of the form F_i^c for some $c > 0$. In this case, we cannot have $i = n$ or $i = n - 1$ because of condition (ii) of Lemma 2.4.4. Suppose for the moment that $i \neq 1$. Then the relations in $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ show that

$$E_i^c E_n F_i^c = E_n E_i^c F_i^c.$$

We can then proceed as in the first case to show that

$$\begin{aligned} E_i^c E_n t_k t_{k-1} \cdots t_1 1_\lambda &= E_n E_i^c t_k 1_\mu t_{k-1} t_{k-2} \cdots t_1 1_\lambda \\ &= E_n z' 1_\mu t_{k-1} t_{k-2} \cdots t_1 1_\lambda \end{aligned}$$

where z' is nonzero. Here we have used Lemma 2.2.7 (ii), which is applicable because M is nonzero and $\mu_i \geq c$. Again, we are done by induction in this case.

The remaining case is the possibility that $t_k = F_1^c$ for some $c > 0$. The relations in $\mathbf{U}(\widehat{\mathfrak{g}}_n)$ show that

$$E_1^c E_n F_1^c = E_1^c F_1^c E_n,$$

and so we have

$$E_1^c E_n t_k t_{k-1} \cdots t_1 1_\lambda = E_1^c t_k E_n 1_\mu t_{k-1} t_{k-2} \cdots t_1 1_\lambda.$$

Because F_1^c arises from a distinguished term, we have $\mu_2 = 0$. By Lemma 2.2.6 (i), we have

$$E_1^c F_1^c E_n 1_\mu = E_1^c F_1^c 1_{\mu'} E_n,$$

where again $\mu'_2 = 0$ (recall that $n \geq 3$). Hence

$$E_1^c E_n t_k t_{k-1} \cdots t_1 1_\lambda = z'' 1_{\mu'} E_n t_{k-1} t_{k-2} \cdots t_1 1_\lambda.$$

Furthermore, z'' is nonzero. To see why, we recall that by condition (ii) of Lemma 2.4.4, there are no occurrences of E_1 or E_n or F_n in M and that by condition (iii), all the occurrences of F_1 occur consecutively. Repeated applications of Lemma 2.2.6 then show that $\mu'_1 = \lambda_1$. Lemma 2.2.7 (ii) then applies again to yield

$$E_1^c F_1^c 1_{\mu'} = z'' 1_{\mu'},$$

and z'' is nonzero because by condition (iv) of Lemma 2.4.4, $c \leq \lambda_1 - 1 < \lambda_1 = \mu'_1$. Once again, the assertion follows by induction in this case.

Finally, we observe that since both M and $\sigma(M)$ avoid occurrences of E_n and F_n , they lie in the subalgebra of T corresponding to $\mathbf{S}_v(n, r)$. This means that M and $\sigma(M)$ lie in $\text{Im}(\iota)$, and the proof follows. \square

Corollary 2.4.6 *If $\lambda \in \Lambda(n, r)$, the elements $E_n 1_\lambda$, $1_\lambda F_n$ and 1_λ lie in the image of ι .*

Proof If $\lambda_1 = 0$, then $E_n 1_\lambda = 1_\lambda F_n = 0$ and the assertion is trivial. Otherwise, the assertion follows by combining lemmas 2.4.3 and 2.4.5. \square

Proof of Theorem 1.6.1 (modulo Lemma 2.4.4) By Lemma 2.4.2 and Corollary 2.4.6, the generators of T listed in Lemma 2.4.1 all lie in $\text{Im}(\iota)$. This proves that ι is surjective, and taken in conjunction with Proposition 2.3.11, we see that ι is an isomorphism. This completes the proof of Theorem 1.6.1 (modulo Lemma 2.4.4). \square

2.5 Proof of Lemma 2.4.4

The only other ingredient needed to prove Theorem 1.6.1 is Lemma 2.4.4.

Definition 2.5.1 Let $\mu, \nu \in \Lambda(n, r)$. We say that μ and ν are *Z-equivalent* if they become equal after their zero parts have been deleted. (In other words, μ and ν correspond to the same parabolic subgroup of the symmetric group.)

Lemma 2.5.2 Let $\lambda \in \Lambda(n, r)$ with $\lambda_1 > 0$.

(i) There exists a nonzero distinguished monomial M_1 in the generators E_2, E_3, \dots, E_{n-1} , such that the occurrences of E_{n-1} occur consecutively, satisfying

$$M_1 = 1_\mu M_1 1_\lambda.$$

Here, $\mu = \mu(\lambda)$ is such that (a) μ and λ are Z-equivalent and (b) $\ell(\mu)$ (see Definition 1.3.1) is of the form

$$\underbrace{(1, 1, \dots, 1)}_{\mu_1}, \underbrace{(2, 2, \dots, 2)}_{\mu_2}, \dots, \underbrace{(k, k, \dots, k)}_{\mu_k}.$$

Furthermore, $\mu_1 = \lambda_1$, $k \leq r$ and thus $\mu_{k+1} = \mu_{k+2} = \dots = \mu_{r+1} = \dots = \mu_n = 0$.

(ii) Let μ be as in part (i) above, and let ν be the unique element of $\Lambda(n, r)$ such that (a) $\nu_a = \mu_{i+1}$ whenever

$$a = 1 + \sum_{j=1}^i \mu_j$$

for any $0 \leq i < r$, and (b) $\nu_a = 0$ for other values of a . (In particular, $\nu_1 = \mu_1 = \lambda_1$, and ν and μ are Z-equivalent.) Then there exists a nonzero distinguished monomial M_2 in the generators F_2, F_3, \dots, F_{n-2} satisfying

$$M_2 = 1_\nu M_2 1_\mu.$$

(iii) Let μ and ν be as in (ii) above. Then for each $1 \leq i \leq r$ we have

$$\sum_{j=1}^{\mu_i} \nu_{b(i)+j} = \mu_i = \nu_{b(i)+1},$$

where $b(i) = \sum_{k < i} \mu_k$. Summing over all i , this yields

$$\sum_{j=1}^r \nu_j = r,$$

and hence $\nu_{r+1} = \nu_{r+2} = \dots = \nu_n = 0$.

(iv) Let ν be as in part (ii) above. Then there exists a nonzero distinguished monomial M_3 in the generators $F_1, F_2, F_3, \dots, F_{n-2}$ satisfying

$$M_3 = 1_\omega M_3 1_\nu.$$

Furthermore, the occurrences of F_1 occur consecutively, and there are $\lambda_1 - 1$ occurrences of F_1 .

Note Note that in the above situation, μ and ν are uniquely determined by λ . For example, if $n = 9$, $r = 7$ and

$$\lambda = (2, 0, 0, 3, 0, 0, 0, 0, 2),$$

then we have

$$\mu = (2, 3, 2, 0, 0, 0, 0, 0, 0),$$

and

$$\nu = (\underbrace{2, 0}_2, \underbrace{3, 0, 0}_3, \underbrace{2, 0}_2, 0, 0).$$

In this case, we could take

$$M_1 = 1_\mu E_3^2 E_4^2 E_5^2 E_6^2 E_7^2 E_8^2 E_2^3 E_3^3 1_\lambda,$$

$$M_2 = 1_\nu F_2^3 F_5^2 F_4^2 F_3^2 1_\mu,$$

$$M_3 = 1_\omega F_1 F_4 F_3^2 F_6 1_\nu.$$

Proof To prove (i), let $\lambda \in \Lambda(n, r)$ be such that $\lambda_i = 0$ and $\lambda_{i+1} = c > 0$. Lemma 2.2.7 (i) implies that $E_i^c 1_\lambda$ is a nonzero element of T , and iterated applications of Lemma 2.2.6(i) show that $E_i^c 1_\lambda = 1_\xi E_i^c$, where

$$\xi_j = \begin{cases} \lambda_{i+1} & \text{if } j = i, \\ \lambda_i & \text{if } j = i + 1, \\ \lambda_j & \text{otherwise.} \end{cases}$$

Repeated applications of this procedure can be used to move all the zero parts of λ to the right. Since $\lambda_1 > 0$ by assumption, we have $\mu_1 = \lambda_1$, and it is never necessary to use an application of E_1 . It is necessary to use applications of E_{n-1} if and only if $\lambda_n > 0$, but in this case they may all be applied consecutively. Since there are at most r nonzero parts in λ_1 and since $n > r$, we have $k \leq r$. We have $M_1 \neq 0$ based on an inductive argument using Lemma 2.2.11. The other assertions of (i) follow.

The claims in (ii) concerning ν are routine apart from the assertion regarding the distinguished monomial. The proof of (iii) then follows from the characterization of ν given in (ii). (The reader may find it helpful here to look at the note preceding this proof.) The entries $\nu_{r+1}, \nu_{r+2}, \dots, \nu_n$ can effectively be ignored for the rest of the proof.

The proof of the last assertion of (ii) follows similar lines to the proof of (i). The main difference is that the aim is to move certain of the zero components of λ to the left. The basic step involves $\mu \in \Lambda(n, r)$ such that $\mu_i = c > 0$ and $\mu_{i+1} = 0$, in which case Lemma 2.2.7 (ii) shows that $F_i^c 1_\mu$ is a nonzero element of T , and iterated applications of Lemma 2.2.6 (ii) show that $F_i^c 1_\mu = 1_\xi F_i^c$, where

$$\xi_j = \begin{cases} \mu_{i+1} & \text{if } j = i, \\ \mu_i & \text{if } j = i + 1, \\ \mu_j & \text{otherwise.} \end{cases}$$

Since $v_1 = \mu_1 = \lambda_1 > 0$ by the definition of v_a , no applications of F_1 are necessary. The fact that $v_n = 0$ shows why no applications of F_{n-1} are necessary. As before, Lemma 2.2.11 shows why $M_2 \neq 0$.

To prove (iv), observe that v can be written as the concatenation of maximal segments of the form

$$(v_i, \underbrace{0, 0, \dots, 0}_{v_i-1}).$$

Let us first deal with the case where $v_1 = r$. Here, the monomial may be given explicitly as

$$M_3 = 1_\omega F_{v_1-1}^1 F_{v_1-2}^2 \cdots F_2^{v_1-2} F_1^{v_1-1} 1_v.$$

This monomial is easily checked to be distinguished. In the general case, there is a monomial in the F_i corresponding to each of the maximal segments mentioned above, and monomials corresponding to distinct segments commute. When these monomials are concatenated and flanked by 1_ω and 1_v , we obtain M_3 . The occurrences of F_1 all correspond to the segment containing v_1 , and the explicit formula above shows that these occurrences are consecutive and that there are $\lambda_1 - 1$ of them. (This number may be zero.) As in the proof of (ii) above, the fact that $\omega_n = 0$ explains why there are no occurrences of F_{n-1} . Also as above, Lemma 2.2.11 shows why $M_3 \neq 0$. \square

Proof of Lemma 2.4.4 In the notation of Lemma 2.5.2, the required monomial is

$$M = 1_\omega M_3 1_v M_2 1_\mu M_1 1_\lambda.$$

Properties (i)–(iv) of Lemma 2.4.4 follow from the various parts of Lemma 2.5.2. Since M_1 , M_2 and M_3 are nonzero, M is nonzero by Lemma 2.2.11.

This completes the proof of Lemma 2.4.4, and therefore of Theorem 1.6.1. \square

2.6 An alternative presentation of $\widehat{\mathbf{S}}_v(n, r)$

Lusztig [16, Part IV] has defined a modified form of a quantized enveloping algebra, by replacing the zero part of the algebra with an infinite system of pairwise orthogonal idempotents, acting on modules as weight space projectors. The modified form has a canonical basis with remarkable properties, similar to properties of the canonical basis of the positive part of the original quantized algebra.

The following presentation of the algebra $\widehat{\mathbf{S}}_v(n, r)$, compatible with Lusztig's modified form of $\mathbf{U}(\widehat{\mathfrak{g}}_n)$, is equivalent with the presentation given in Theorem 1.6.1.

Theorem 2.6.1 *Assume that $n > r$. Over $\mathbb{Q}(v)$, the algebra $\widehat{\mathbf{S}}_v(n, r)$ is isomorphic with the associative algebra (with 1) given by the generators i_λ ($\lambda \in \Lambda(n, r)$), E_i, F_i ($1 \leq i \leq n$) and relations*

$$i_\lambda i_\mu = \delta_{\lambda, \mu} i_\lambda; \quad \sum_{\lambda \in \Lambda(n, r)} i_\lambda = 1; \quad (\text{R1})$$

$$E_i \mathfrak{i}_\lambda = \begin{cases} \mathfrak{i}_{\lambda+\alpha_i} E_i & \text{if } \lambda_{i+1} > 0, \\ 0 & \text{otherwise;} \end{cases} \quad (\text{R2})$$

$$F_i \mathfrak{i}_\lambda = \begin{cases} \mathfrak{i}_{\lambda-\alpha_i} F_i & \text{if } \lambda_i > 0, \\ 0 & \text{otherwise;} \end{cases} \quad (\text{R3})$$

$$E_i F_j - F_j E_i = \delta_{i,j} \sum_{\lambda \in \Lambda(n,r)} [\lambda_j - \lambda_{j+1}] \mathfrak{i}_\lambda \quad (\text{R4})$$

along with relations (Q6)–(Q9) of Definition 1.4.1. Here we regard weights as infinite periodic sequences, as in Section 2.2 above.

Proof Let A be the algebra defined by the presentation of the theorem, and let T be the algebra defined by the presentation of Theorem 1.6.1.

By Definition 2.2.2, Lemma 2.2.4, and Lemma 2.2.6 the elements 1_λ ($\lambda \in \Lambda(n, r)$), E_i, F_i ($1 \leq i \leq n$) of T satisfy the relations (R1)–(R3). The E_i, F_i ($1 \leq i \leq n$) already satisfy relations (Q6)–(Q9) by assumption. By applying Remark 1.6.4 to the results in [7, Theorem 3.4] we see that (R4) holds as well for all $1 \leq i, j < n$. If one or both of i, j is equal to n , then we choose a different embedding of $\mathbf{S}_v(n, r)$ in T , one which includes the values of i, j in question, and again apply Remark 1.6.4 to the results in [7, Theorem 3.4] to see that (R4) holds in that case as well. In T we have by Lemma 2.2.4 the equalities

$$K_i = K_i \sum_{\lambda} 1_\lambda = \sum_{\lambda} v^{\lambda_i} 1_\lambda; \quad K_i^{-1} = K_i^{-1} \sum_{\lambda} 1_\lambda = \sum_{\lambda} v^{-\lambda_i} 1_\lambda \quad (*)$$

for any $1 \leq i \leq n$, where the sums are taken over all $\lambda \in \Lambda(n, r)$. Hence, the elements 1_λ ($\lambda \in \Lambda(n, r)$), E_i, F_i ($1 \leq i \leq n$) generate T , and the map

$$\mathfrak{i}_\lambda \rightarrow 1_\lambda, \quad E_i \rightarrow E_i, \quad F_i \rightarrow F_i$$

defines a surjective quotient map from A onto T .

On the other hand, in the algebra A one defines elements $K_i = \sum_{\lambda} v^{\lambda_i} \mathfrak{i}_\lambda$, $K_i^{-1} = \sum_{\lambda} v^{-\lambda_i} \mathfrak{i}_\lambda$. By following the same line of argument as in the proof of [7, Theorem 3.4], these elements, along with the elements E_i, F_i for $1 \leq i < n$, satisfy the defining relations (Q1)–(Q9), (Q15), (Q16) of T . It remains to show that the elements E_n, F_n also satisfy those relations (along with the K_i, K_i^{-1}). Only relations (Q3), (Q4), and (Q5) are in question since the other relations either hold by assumption or do not involve the elements E_n, F_n .

We now verify that relation (Q3) holds for E_n . By definition of K_i we have

$$K_i E_n = \sum_{\lambda} v^{\lambda_i} \mathfrak{i}_\lambda E_n$$

and by relation (R2) this takes the form

$$K_i E_n = \sum_{\lambda} v^{\lambda_i} E_n \mathfrak{i}_{\lambda-\alpha_n}$$

where, for convenience of notation, we take both sums over the set of all $\lambda \in \mathbb{Z}^n$ satisfying $\sum \lambda_i = r$, with the understanding that \mathfrak{i}_λ is interpreted to be 0 in case

any part of λ is negative. (This makes all the sums in question finite.) Now replace $\lambda - \alpha_n$ by $\mu \in \mathbb{Z}^n$ and the above gives relation (Q3) for E_n .

The proof that relation (Q4) holds for F_n is similar.

Finally, we verify that (Q5) holds. By the given relation (R4) we have

$$E_i F_j - F_j E_i = \delta_{ij} \sum_{\lambda} [\lambda_j - \lambda_{j+1}] \mathbf{i}_{\lambda}$$

and this gives 0 unless $i = j$, so (Q5) holds in case $i \neq j$. Assuming that $i = j$, the above sum becomes

$$\begin{aligned} E_i F_i - F_i E_i &= \sum_{\lambda} \frac{v^{\lambda_i - \lambda_{i+1}} - v^{-\lambda_i + \lambda_{i+1}}}{v - v^{-1}} \mathbf{i}_{\lambda} \\ &= \frac{(\sum_{\lambda} v^{\lambda_i} \mathbf{i}_{\lambda} v^{-\lambda_{i+1}} \mathbf{i}_{\lambda}) - (\sum_{\lambda} v^{-\lambda_i} \mathbf{i}_{\lambda} v^{\lambda_{i+1}} \mathbf{i}_{\lambda})}{v - v^{-1}} \\ &= \frac{(\sum_{\lambda} v^{\lambda_i} \mathbf{i}_{\lambda})(\sum_{\lambda} v^{-\lambda_{i+1}} \mathbf{i}_{\lambda}) - (\sum_{\lambda} v^{-\lambda_i} \mathbf{i}_{\lambda})(\sum_{\lambda} v^{\lambda_{i+1}} \mathbf{i}_{\lambda})}{v - v^{-1}} \end{aligned}$$

using the orthogonality of the system of idempotents. By the definition of the K_i, K_i^{-1} this proves (Q5) in case $i = j$.

We claim that the elements K_i, K_i^{-1}, E_i, F_i (for $1 \leq i \leq n$) generate A . To see this, it suffices to show that the K_i, K_i^{-1} generate the zero part of A (the span of the \mathbf{i}_{λ}). From the definition of K_i and K_i^{-1} it follows that

$$K_j \mathbf{i}_{\lambda} = v^{\lambda_j} \mathbf{i}_{\lambda}; \quad K_j^{-1} \mathbf{i}_{\lambda} = v^{-\lambda_j} \mathbf{i}_{\lambda}$$

and thus $K_j = K_j \sum_{\lambda} \mathbf{i}_{\lambda} = \sum_{\lambda} v^{\lambda_j} \mathbf{i}_{\lambda}$ and $K_j^{-1} = K_j^{-1} \sum_{\lambda} \mathbf{i}_{\lambda} = \sum_{\lambda} v^{-\lambda_j} \mathbf{i}_{\lambda}$, where the sums are over all $\lambda \in \Lambda(n, r)$. Hence it follows that

$$\begin{aligned} \begin{bmatrix} K_j \\ t \end{bmatrix} &= \prod_{s=1}^t \frac{K_j v^{-s+1} - K_j^{-1} v^{s-1}}{v^s - v^{-s}} \\ &= \prod_{s=1}^t \frac{(\sum_{\lambda} v^{\lambda_j - s + 1} \mathbf{i}_{\lambda}) - (\sum_{\lambda} v^{-\lambda_j + s - 1} \mathbf{i}_{\lambda})}{v^s - v^{-s}} \\ &= \prod_{s=1}^t \sum_{\lambda} \frac{v^{\lambda_j - s + 1} - v^{-\lambda_j + s - 1}}{v^s - v^{-s}} \mathbf{i}_{\lambda} \\ &= \sum_{\lambda} \prod_{s=1}^t \frac{v^{\lambda_j - s + 1} - v^{-\lambda_j + s - 1}}{v^s - v^{-s}} \mathbf{i}_{\lambda} \\ &= \sum_{\lambda} \begin{bmatrix} \lambda_j \\ t \end{bmatrix} \mathbf{i}_{\lambda} \end{aligned}$$

where we have again made use of the orthogonality of the idempotents to interchange the product and sum. From this and the orthogonality of idempotents it follows that for any $\mu \in \Lambda(n, r)$ we have

$$\begin{bmatrix} K_1 \\ \mu_1 \end{bmatrix} \cdots \begin{bmatrix} K_n \\ \mu_n \end{bmatrix} = \prod_{j=1}^n \left(\sum_{\lambda} \begin{bmatrix} \lambda_j \\ \mu_j \end{bmatrix} \mathbf{i}_{\lambda} \right)$$

$$= \sum_{\lambda} \begin{bmatrix} \lambda_1 \\ \mu_1 \end{bmatrix} \cdots \begin{bmatrix} \lambda_n \\ \mu_n \end{bmatrix} \dot{\iota}_{\lambda}$$

where λ runs over the set $\Lambda(n, r)$ in the sums. The only non-zero term in the last sum is when $\lambda = \mu$, so

$$\begin{bmatrix} K_1 \\ \mu_1 \end{bmatrix} \cdots \begin{bmatrix} K_n \\ \mu_n \end{bmatrix} = \dot{\iota}_{\mu}.$$

This proves the claim. (The reader should refer to [16, §1.3] for definitions and basic properties of quantized binomial coefficients used here.)

We have shown that the elements K_i, K_i^{-1}, E_i, F_i (for $1 \leq i \leq n$) generate the algebra A , and moreover satisfy all the defining relations for the algebra T . It follows that the map

$$K_i^{\pm 1} \rightarrow K_i^{\pm 1}, \quad E_i \rightarrow E_i, \quad F_i \rightarrow F_i$$

is a surjective quotient map from T onto A .

Now consider the composite surjective map $T \rightarrow A \rightarrow T$. This is the clearly identity on E_i, F_i . Moreover, by equations (*) above the composite map takes K_i to $\sum_{\lambda} v^{\lambda_i} 1_{\lambda} = K_i$. Similarly, it takes K_i^{-1} to itself. Thus the composite is the identity, and thus each quotient map $T \rightarrow A$ and $A \rightarrow T$ is an algebra isomorphism. \square

3 The classical case

All of the results of this paper have analogues in the case $v = 1$. The proofs run parallel to the arguments given here, but are often easier. We will outline the main results here, and leave it to the reader to fill in the details.

3.1 The affine Schur algebra

The analogue of Definition 1.3.4 is the following

Definition 3.1.1 The affine Schur algebra $\widehat{S}(n, r)$ over \mathbb{Z} is defined by

$$\widehat{S}(n, r) := \text{End}_{\widehat{W}} \left(\bigoplus_{\lambda \in \Lambda(n, r)} x_{\lambda} \widehat{W} \right),$$

where $x_{\lambda} = \sum_{w \in \widehat{W}_{\lambda}} w$.

There is a basis of $\widehat{S}(n, r)$ similar to the basis of $\widehat{S}_q(n, r)$ given in Theorem 1.3.6. The details are left to the reader.

Definition 3.1.2 The associative, unital algebra $U(\widehat{\mathfrak{gl}}_n)$ over \mathbb{Q} is given by generators

$$e_i, f_i, H_i \quad (1 \leq i \leq n)$$

subject to the relations

$$H_i H_j = H_j H_i; \quad (\text{q1})$$

$$H_i e_j - e_j H_i = \varepsilon^+(i, j) e_j; \quad (\text{q2})$$

$$H_i f_j - f_j H_i = \varepsilon^-(i, j) f_j; \quad (\text{q3})$$

$$e_i f_j - f_j e_i = \delta_{ij} (H_j - H_{j+1}); \quad (\text{q4})$$

$$e_i e_j = e_j e_i \text{ if } i \text{ and } j \text{ are not adjacent}; \quad (\text{q5})$$

$$f_i f_j = f_j f_i \text{ if } i \text{ and } j \text{ are not adjacent}; \quad (\text{q6})$$

$$e_i^2 e_j - 2e_i e_j e_i + e_j e_i^2 = 0 \text{ if } i \text{ and } j \text{ are adjacent}; \quad (\text{q7})$$

$$f_i^2 f_j - 2f_i f_j f_i + f_j f_i^2 = 0 \text{ if } i \text{ and } j \text{ are adjacent}. \quad (\text{q8})$$

As in Definition 1.4.3 the notion of adjacency takes place in the Dynkin diagram of type \widehat{A}_{n-1} , so we read indices modulo n in this definition.

This algebra is a Hopf algebra in a natural way, and the quotient of $U(\widehat{\mathfrak{gl}}_n)$ by the kernel of its action on a suitably defined tensor space is isomorphic as a \mathbb{Q} -algebra to $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{S}(n, r)$.

3.2 Main results

The analogue of Theorem 1.6.1 is the

Theorem 3.2.1 *Let $n > r$, and identify $\widehat{S}(n, r)$ with the quotient of $U(\widehat{\mathfrak{gl}}_n)$ acting on tensor space. Over \mathbb{Q} , the affine Schur algebra $\widehat{S}(n, r)$ is given by generators e_i, f_i, H_i ($1 \leq i \leq n$) subject to relations (q1) to (q8) of Definition 3.1.2 (reading indices modulo n), together with the relations*

$$H_1 + \cdots + H_n = r; \quad (\text{q9})$$

$$H_i(H_i - 1)(H_i - 2) \cdots (H_i - r) = 0. \quad (\text{q10})$$

There is also an equivalent version in terms of idempotents, analogous to Theorem 2.6.1, which we now state.

Theorem 3.2.2 *Assume that $n > r$. Over \mathbb{Q} , the algebra $\widehat{S}(n, r)$ is isomorphic with the associative algebra (with 1) given by the generators \mathfrak{i}_λ ($\lambda \in \Lambda(n, r)$), e_i, f_i ($1 \leq i \leq n$) and relations*

$$\mathfrak{i}_\lambda \mathfrak{i}_\mu = \delta_{\lambda, \mu} \mathfrak{i}_\lambda; \quad \sum_{\lambda \in \Lambda(n, r)} \mathfrak{i}_\lambda = 1; \quad (\text{r1})$$

$$e_i \mathfrak{i}_\lambda = \begin{cases} \mathfrak{i}_{\lambda + \alpha_i} e_i & \text{if } \lambda_{i+1} > 0, \\ 0 & \text{otherwise;} \end{cases} \quad (\text{r2})$$

$$f_i \mathfrak{i}_\lambda = \begin{cases} \mathfrak{i}_{\lambda - \alpha_i} f_i & \text{if } \lambda_i > 0, \\ 0 & \text{otherwise;} \end{cases} \quad (\text{r3})$$

$$e_i f_j - f_j e_i = \delta_{i,j} \sum_{\lambda \in \Lambda(n,r)} (\lambda_j - \lambda_{j+1}) \mathbf{i}_\lambda \quad (\text{r4})$$

along with relations (q5)–(q8) of Definition 3.1.2. Here we regard weights as infinite periodic sequences, as in Section 2.2 above.

These relations are obtained from those in Theorem 2.6.1 by setting $v = 1$.

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