MATH 8174: Assignment 14

1. If $\{\beta - s\alpha, \dots, \beta, \dots, \beta + t\alpha\}$ is the α-string through β, show that $\beta + r\alpha$ is a root if and only if $-s \le r \le t$.

   Here are some suggested steps; recall that the "if" direction was proved in class.

(1) By considering the $-\alpha$-string through $-\beta$, show that without loss of generality, we may suppose for a contradiction that $\beta + r\alpha$ is a root for some $r \ge t + 2$.

(2) Choose this r to be as small as possible and consider the α-string through $\beta + r\alpha$; this has the form

$$\beta + r\alpha, \dots, \beta + (r + m)\alpha$$

for some $m \ge 0$. Use the calculation of (7.9) to show that

$$\beta(x) + r\alpha(x) = -\frac{m}{2}\alpha(x).$$

(3) Use (9.2)(f) to derive a contradiction from the equation

$$\frac{(s-t)}{2}\alpha(x) = \left(-\frac{m}{2} - r\right)\alpha(x).$$

2. Show that, in the standard notation for a semisimple Lie algebra, the elements e_α, $e_{-\alpha}$ and h_α span a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

3. The roots of a semisimple Lie algebra come in pairs $\{\alpha, -\alpha\}$. Show that we have $\dim(H) \le s$, where H is the Cartan subalgebra. Give an example of a semisimple Lie algebra L for which $\dim(H) = s$.

4. Let H be a vector space over a field k, and let $(\, , \,)$ be a symmetric bilinear form on H. If $\{h_1, \dots, h_l\}$ is a basis for H and if $a_{ij} = (h_i, h_j)$, prove that $(\, , \,)$ is nondegenerate if and only if (a_{ij}) is a nonsingular matrix.