MATH 6140: Final examination. Monday, 7 May 2007.

Put **your name** on each answer sheet. Answer **all three** questions.

Show your working in full. Formula sheets, calculators, notes and books are not permitted.

1. Let V be an n-dimensional vector space over a field F (where $n \geq 1$). Quoting any standard results that you need, find the dimension of the exterior algebra $\bigwedge(V)$. Is the symmetric algebra $\mathcal{S}(V)$ finite or infinite dimensional?

2. A *normal basis* for a Galois extension E/F is an F-basis for E of the form

$$\{\sigma(\alpha) : \sigma \in \mathrm{Gal}(E/F)\};$$

in other words, it consists of a certain element $\alpha \in E$ together with all of its Galois conjugates. The Normal Basis Theorem states that every Galois extension (of finite degree) has a normal basis. Let E be a field with p^n elements and let F be its prime subfield (with p elements); you may assume that E/F is Galois.

(i) What is meant by the Frobenius automorphism of E? What is the structure of the Galois group $\mathrm{Gal}(E/F)$?

(ii) Using the Normal Basis Theorem mentioned above, give the rational canonical form of the Frobenius automorphism of E, regarded as an F-linear map from E to E.

(iii) Assume further that n is not a multiple of p. Give the Jordan canonical form (over a field containing all the eigenvalues) of the Frobenius automorphism of E. (**For extra credit:** Give the Jordan canonical form in the case where p divides n.)

3. Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ has degree 2 over $\mathbb{Q}(\sqrt{2})$. Hence, or otherwise, prove that $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, i)$ is a Galois extension of $\mathbb{Q}$ of degree 8. Find a $\mathbb{Q}$-basis of K and describe the group $\mathrm{Gal}(K/\mathbb{Q})$ explicitly. Without calculating any minimal polynomials, prove that

$$K = \mathbb{Q}(\sqrt{2} + \sqrt{3} + i).$$
