RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra
Ph.D. Preliminary Exam

January 2011

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

3. There are 6 problems, each worth the same number of points. Please do them all.

Algebra Prelim **January 2011**

1. Find explicit generators for subgroups of the symmetric group S_7 that are isomorphic to each of the groups (a) $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, (b) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and (c) D_8, the dihedral group of order 8. Show also that S_7 has no subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z}$ or to Q_8.

2. Classify the groups of order 20, and deduce that, up to isomorphism, there is a unique group of order 20 with no element of order 10.

3. Let R be a commutative Noetherian ring with $1 \neq 0$. (Recall that this means that any ascending chain

$$I_1 \subset I_2 \subset \cdots$$

of ideals of R is eventually stationary, i.e., $I_n = I_{n+1} = \ldots$ for some integer $n \geq 1$.) If I ($\neq R$) is an ideal of R, show that there are prime ideals $P_1, \ldots, P_n$ ($n \geq 1$, an integer) of R such that each P_i contains I and $P_1 P_2 \cdots P_n \subset I$.

4. Let F be a field, and let $A = (a_{ij}) \in M_n(F)$ be the upper triangular matrix for which $a_{ij} = 1$ if $j \geq i$, and $a_{ij} = 0$ otherwise. Find the Jordan canonical form and rational canonical form of A. Prove that the n matrices $I = A^0,\ A = A^1,\ A^2,\ A^3,\ \ldots,\ A^{n-1}$ are linearly independent over F.

5. Let $\mathbb{F}_2$ be the field with two elements, and let $F = \overline{\mathbb{F}_2}$ be the algebraic closure of $\mathbb{F}_2$. Show that every nonzero element of F is a root of unity, and find a formula (in terms of $n \in \mathbb{N}$) for the number of elements of F whose multiplicative order is exactly n.

6. Let $\zeta = e^{\frac{2\pi\sqrt{-1}}{12}}$ and $K = \mathbb{Q}(\zeta) \subset \mathbb{C}$.

(i) Find the Galois group of $K/\mathbb{Q}$.

(ii) Use the result of (i) to determine all the subfields of K.