MATH 6130: Final examination. Wednesday, December 20 2023.

Put **your name** on each answer sheet. Answer **all three** questions.

Justify all your answers. Formula sheets, calculators, notes and books are not permitted.

1. Let C_n be the cyclic group of order n written multiplicatively, let S_n be the symmetric group on n letters, let D_{2n} be the dihedral group of order $2n$, and define $K = C_2$. Let $\alpha_1, \alpha_2, \alpha_3$ be the automorphisms of the group $H = C_3 \times C_3$ given by

$$\alpha_1 : (a, b) \mapsto (a, b), \quad \alpha_2 : (a, b) \mapsto (a, b^{-1}), \quad \text{and} \quad \alpha_3 : (a, b) \mapsto (a^{-1}, b^{-1}).$$

(i) Use the automorphisms α_i to construct three pairwise nonisomorphic semidirect products $H \rtimes K$. [You may assume that the α_i are indeed automorphisms.]

(ii) Use semidirect products to construct five pairwise nonisomorphic groups of order 18.

(iii) It turns out that there are only five groups of order 18 up to isomorphism, namely C_{18}, $C_3 \times C_6$, $C_3 \times S_3$, D_{18}, and a "mystery" group E. Match these five groups with the groups constructed in (ii).

(iv) Determine which groups of order 18 are (a) nilpotent and/or (b) solvable.

2. Let $\phi_{\sqrt[3]{2}} : \mathbb{Q}[x] \longrightarrow \mathbb{R}$ be the evaluation homomorphism at $\sqrt[3]{2} \in \mathbb{R}$.

(i) Prove that the kernel of $\phi_{\sqrt[3]{2}}$ is precisely $\langle x^3 - 2 \rangle$.

(ii) Prove carefully that $\mathbb{Q}[\sqrt[3]{2}] = \{p + q\sqrt[3]{2} + r\sqrt[3]{4} : p, q, r \in \mathbb{Q}\}$ is a subfield of $\mathbb{R}$. [Hint: use the First Isomorphism Theorem.]

(iii) You may assume that $\mathbb{Z}[\sqrt[3]{2}] = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Z}\}$ is a subring of $\mathbb{Q}[\sqrt[3]{2}]$. Show that $1 + \sqrt[3]{2} + \sqrt[3]{4}$ is a unit in $\mathbb{Z}[\sqrt[3]{2}]$. [Hint: $(1 + x + x^2)(x - 1) - (x^3 - 2) = 1$.]

(iv) Show that $\mathbb{Z}[\sqrt[3]{2}]$ has infinitely many units.

3. Factorize the polynomial $x^4 + 3$ into irreducibles in the following rings, and prove that the factors are irreducible:

(i) $\mathbb{Z}_7[x] = (\mathbb{Z}/7\mathbb{Z})[x]$; (ii) $\mathbb{Z}[x]$; (iii) $\mathbb{Q}[x]$; (iv) $\mathbb{R}[x]$; (v) $\mathbb{C}[x]$.