MATH 6130: First midterm examination. Wednesday, 27 September 2023.

Put **your name** on each answer sheet. Answer **all three** questions.

Show all your work.

Formula sheets, calculators, notes and books are not permitted.

1. Let $G = \mathbb{R}$ be the group of real numbers under addition, and let $H = \mathbb{C}^*$ be the group of nonzero complex numbers under multiplication. Let $\phi : G \to H$ be the exponentiation function $\phi(r) = e^{2\pi i r}$.

(i) Prove that ϕ is a homomorphism of groups and find its image and kernel.

(ii) Using standard properties and the result of (i), or otherwise, prove that the subset

$$S = \{a + bi : a^2 + b^2 = 1\}$$

of H is a subgroup of H.

2. Recall that $GL_n(\mathbb{R})$ is the group of invertible $n \times n$ matrices with real entries, under matrix multiplication. Define the subset $O_n(\mathbb{R})$ of $GL_n(\mathbb{R})$ by

$$O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid A^T = A^{-1}\},$$

where A^T denotes the transpose of A. Prove that $O_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$. [The group $O_n(\mathbb{R})$ is called the *orthogonal group*.]

3. Let G be a group and let H be a subgroup of G.

(i) Prove that there is a well-defined action of G on the left cosets of H in G given by

$$g \,.\, xH = gxH,$$

and find the stabilizer in G of the coset H.

(ii) Recall from class that the symmetric group S_4 has a subgroup D of order 8. [You do not need to know anything else about D for this question.] By setting $G = S_4$ and $H = D$ in part (i), show that there is a homomorphism of groups $\phi : S_4 \longrightarrow S_3$ such that $\ker \phi \leq D$.