1. (40) The Weyl group of type E_7, denoted by $W(E_7)$, is a group of order 2903040. You may assume the following facts about $W(E_7)$: (a) the center, $Z(W(E_7))$, of $W(E_7)$ has order 2; (b) $W(E_7)$ has a subgroup called $\mathrm{Sp}(6,2)$ of order 1451520; and (c) $\mathrm{Sp}(6,2)$ is a simple group.

(i) Prove that $\mathrm{Sp}(6,2)$ is nonabelian. Deduce that $W(E_7)$ is nonabelian and that the commutator subgroup of $W(E_7)$ is not the trivial group.

(ii) Prove that $Z(W(E_7))$ is not a subgroup of $\mathrm{Sp}(6,2)$, and deduce that the intersection $Z(W(E_7)) \cap \mathrm{Sp}(6,2)$ is trivial.

(iii) Prove that $\mathrm{Sp}(6,2)$ is normal in $W(E_7)$.

(iv) Prove that $\mathrm{Sp}(6,2)$ is the commutator subgroup of $W(E_7)$.

2. (40) Let S_4 be the symmetric group on 4 symbols, of order 24. You may assume without proof that S_4 has a subgroup $S_3 \leq S_4$ of order 6, and a normal subgroup $V_4 \trianglelefteq S_4$ of order 4, whose elements are given as follows:

$$S_3 = \{\mathrm{id}, (12), (23), (13), (123), (132)\},$$

$$V_4 = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}.$$

(i) State the Second Isomorphism Theorem for groups, which applies to a group G with a subgroup H and a normal subgroup N.

(ii) Let S_3V_4 be the subset of S_4 given by

$$S_3V_4 = \{hn : h \in S_3,\ n \in V_4\}.$$

Explain briefly why S_3V_4 is a subgroup of S_4.

(iii) Use the Second Isomorphism Theorem to prove that the group $\dfrac{S_3V_4}{V_4}$ is a nonabelian group of order 6.

(iv) Use your answer to (iii) to show that the subgroup S_3V_4 is the whole of S_4, and that every element $g \in S_4$ can be expressed *uniquely* in the form hn for some $h \in S_3$ and $n \in V_4$.

(v) Use your answer to (iii) to show that V_4 is not the commutator subgroup of S_4.

3. (20) Let E be a field (of any characteristic) and let $f(x) = x^4 - x^2 + 1$ be an element of the polynomial ring $E[x]$. Assume that the polynomial $f(x)$ has a root, α, in the field E. Let $\phi_\alpha : E[x] \to E$ be the evaluation homomorphism at α; in other words, $\phi_\alpha(f(x)) = f(\alpha)$.

(i) Verify that the identity

$$(x^4 - x^2 + 1)(x^2 + 1) = x^6 + 1$$

holds in $E[x]$.

(ii) By applying ϕ_α to both sides of the equation in (i), or otherwise, show that the root α of $f(x)$ satisfies $\alpha^6 = -1$.

(iii) Using your answer to (ii), verify that each of α^5, α^7 and α^{11} is a root of $f(x)$ in E.

4. (50) Let $f(x) = x^4 - x^2 + 1$. Find a factorization of $f(x)$ into irreducible polynomials in each of the following rings, justifying your answers briefly:

(i) $\mathbb{Z}_2[x]$;

(ii) $\mathbb{Z}_{13}[x]$ (hint: look for a root and use the result of Question 3);

(iii) $\mathbb{Q}[x]$ (hint: use Gauss' Lemma (Theorem 23.11));

(iv) $\mathbb{R}[x]$ (hint: use your answer to (iii) or look for a factor of the form $x^2 + ax + 1$);

(v) $\mathbb{C}[x]$.

5. (50) In this question, the symbols $\sqrt[3]{2}$ and $\sqrt[3]{4}$ refer to the real cube roots of 2 and 4, respectively. Let $\phi_{\sqrt[3]{2}} : \mathbb{Q}[x] \to \mathbb{R}$ denote the evaluation homomorphism at $\sqrt[3]{2}$; you may assume that $\phi_{\sqrt[3]{2}}$ is a homomorphism of rings.

(i) Prove that the polynomial $x^3 - 2$ is irreducible in $\mathbb{Q}[x]$, and that the principal ideal $\langle x^3 - 2\rangle$ is maximal in $\mathbb{Q}[x]$.

(ii) Prove that any polynomial $f(x) \in \mathbb{Q}[x]$ can be expressed in the form

$$f(x) = a + bx + cx^2 + (x^3 - 2)g(x),$$

where $a, b, c \in \mathbb{Q}$ and $g(x) \in \mathbb{Q}[x]$. (It is not enough to name a theorem; you need to explain exactly how you are using it.)

(iii) Use your answer to (ii) to prove that $\phi_{\sqrt[3]{2}}(f(x)) = a + b\sqrt[3]{2} + c\sqrt[3]{4}$.

(iv) Let D be the subset of $\mathbb{R}$ given by

$$D = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Q}\}.$$

Prove that D is a subring of $\mathbb{R}$.

(v) Prove that the ideal $\langle x^3 - 2\rangle$ is contained in the kernel, $\ker(\phi_{\sqrt[3]{2}})$, of $\phi_{\sqrt[3]{2}}$. Use your answer to (i) to show that in fact we have $\langle x^3 - 2\rangle = \ker(\phi_{\sqrt[3]{2}})$.

(vi) State the First Isomorphism Theorem for rings, which applies to a homomorphism of rings $\phi : R \to S$.

(vii) By applying the First Isomorphism Theorem to the homomorphism $\phi_{\sqrt[3]{2}} : \mathbb{Q}[x] \to \mathbb{R}$, prove that the ring D in (iv) is in fact a field, and find the characteristic of D.

Name: ______________________________

University of Colorado

Mathematics 3140: Final Exam

July 5, 2019

Problem	Points	Score
1	40	
2	40	
3	20	
4	50	
5	50	
Total	200	