1. (40) Let $S = \mathbb{R}\backslash\{0\}$ be the set of all nonzero real numbers, both positive and negative. Define an operation $*$ (given in terms of multiplication of real numbers) by

$$a * b := 3ab.$$

(i) Show that $*$ is a binary operation on $S$, and that $(S, *)$ is an abelian group.

(ii) Show that the set $H = \{3^r : r \in \mathbb{Z}\}$ consisting of all integer powers of 3 forms a subgroup of $S$.

2. (20) Let $(\mathbb{Z}, +)$ be the additive group of integers, and let $(H, *) = \{3^r : r \in \mathbb{Z}\}$ be the group given in Question 1 (ii). Let $\phi : \mathbb{Z} \to H$ be given by

$$\phi(a) = 3^{a-1}.$$

(i) Show that $\phi$ is an isomorphism of groups. (You may assume any standard facts about logarithms in base 3 if it is clear how you are using them.)

(ii) Show that $(H, *)$ is cyclic.

(iii) Show that the group $(S, *)$ from Question 1 is not cyclic.

3. (10) List the elements in the subgroup of $\mathbb{Z}_{22}$ (integers modulo 22 under addition modulo 22) generated by the subset $\{6, 14\}$.

4. (10)

(i) Find the order of the symmetric group on 8 letters, $S_8$. (You may leave your answer as a formula.)

(ii) There exists a nonabelian group $Q_8$ of order 8 that is not isomorphic to the dihedral group $D_4$ of order 8. Explain why $S_8$ must have a subgroup isomorphic to $Q_8$.

5. (20) True or False. Mark with a "T" or an "F," and provide a brief explanation (a couple of lines), for each part.

(i) _______________If $(G, *)$ is a group and there exist elements $a, b, c$ of $G$ with $a * c = b * c$, then it must be the case that $a = b$.

(ii) _______________Every element of the group $(\mathbb{Z}_{10}, +_{10})$, with the exception of the identity element, is a generator for $\mathbb{Z}_{10}$.

(iii) _______________There exists a nonabelian group of order 4.

(iv) ________________ There exists a nonabelian group of order 26.

(v) ________________ A subgroup of the symmetric group $S_8$ must be nonabelian.

(vi) ________________ A subgroup of a cyclic group must be cyclic.

**Name:** ______________________

University of Colorado

Mathematics 3140: First In-Class Exam

June 11, 2019

| Problem | Points | Score |
|---|---|---|
| 1 | 40 | |
| 2 | 20 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 20 | |
| Total | 100 | |