Math 6240 First Exam Solution

1. (a) Give the definition of a general connection operator $(X, Y) \mapsto \nabla_X Y$ on a manifold. (In other words, in addition to bilinearity, how must it depend on the vector fields X and Y?)

   Solution: Tensorial in X and differential in Y; that is, $\nabla_{fX} Y = f\nabla_X Y$ and $\nabla_X(gY) = X(g)Y + g\nabla_X Y$ for every function f, g, and every vector field X, Y.

   (b) What two additional conditions are required to obtain a metric (Levi-Civita) connection? Write them explicitly.

   Solution: The torsion-free condition

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

   and the metric compatibility condition

$$X\big(\langle Y, Z\rangle\big) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

   for all vector fields X, Y, and Z.

   (c) Derive the Koszul formula for the Levi-Civita connection $\langle \nabla_X Y, Z\rangle$.

   Solution:

$$\begin{aligned}
\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle - \langle Y, \nabla_X Z\rangle \\
&= X\langle Y, Z\rangle - \langle Y, [X, Z]\rangle - \langle Y, \nabla_Z X\rangle \\
&= X\langle Y, Z\rangle - \langle Y, [X, Z]\rangle - Z\langle Y, X\rangle + \langle \nabla_Z Y, X\rangle \\
&= X\langle Y, Z\rangle - \langle Y, [X, Z]\rangle - Z\langle Y, X\rangle + \langle [Z, Y], X\rangle + \langle \nabla_Y Z, X\rangle \\
&= X\langle Y, Z\rangle - \langle Y, [X, Z]\rangle - Z\langle Y, X\rangle + \langle [Z, Y], X\rangle + Y\langle Z, X\rangle - \langle Z, \nabla_Y X\rangle \\
&= X\langle Y, Z\rangle - \langle Y, [X, Z]\rangle - Z\langle Y, X\rangle + \langle [Z, Y], X\rangle + Y\langle Z, X\rangle - \langle Z, [Y, X]\rangle - \langle Z, \nabla_X Y\rangle
\end{aligned}$$

   Therefore,

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\Big(X\langle Y, Z\rangle - \langle Y, [X, Z]\rangle - Z\langle Y, X\rangle + \langle [Z, Y], X\rangle + Y\langle Z, X\rangle - \langle Z, [Y, X]\rangle\Big).$$

   (d) Verify that the Koszul formula actually does satisfy the definition of a connection, i.e., check the tensoriality properties.

   Solution: First check tensoriality in X:

$$\begin{aligned}
\langle \nabla_{fX} Y, Z\rangle &= \frac{1}{2}\Big(fX\langle Y, Z\rangle - \langle Y, [fX, Z]\rangle - Z\langle Y, fX\rangle + \langle [Z, Y], fX\rangle + Y\langle Z, fX\rangle - \langle Z, [Y, fX]\rangle\Big) \\
&= \frac{1}{2}\Big(fX\langle Y, Z\rangle - f\langle Y, [X, Z]\rangle + Z(f)\langle Y, X\rangle - fZ\langle Y, X\rangle - Z(f)\langle Y, X\rangle + f\langle [Z, Y], X\rangle \\
&\qquad + Y(f)\langle Z, X\rangle - f\langle Z, [Y, X]\rangle - Y(f)\langle Z, X\rangle\Big) \\
&= f\langle \nabla_X Y, Z\rangle.
\end{aligned}$$

Now check that it is a derivation in Y:

$$\begin{aligned}
\langle \nabla_X(gY), Z\rangle &= \frac{1}{2}\Big(X\langle gY, Z\rangle - \langle gY, [X,Z]\rangle - Z\langle gY, X\rangle + \langle [Z, gY], X\rangle + gY\langle Z, X\rangle - \langle Z, [gY, X]\rangle\Big) \\
&= \frac{1}{2}\Big(X(g)\langle Y, Z\rangle + gX\langle Y, Z\rangle - g\langle Y, [X,Z]\rangle - Z(g)\langle Y, X\rangle + Z(g)\langle Y, X\rangle + g\langle [Z,Y], X\rangle \\
&\qquad + gY\langle Z, X\rangle + X(g)\langle Z, Y\rangle - g\langle Z, [Y, X]\rangle\Big) \\
&= X(g)\langle Y, Z\rangle + g\langle \nabla_X Y, Z\rangle \\
&= \langle X(g)Y + g\nabla_X Y, Z\rangle.
\end{aligned}$$

We can also check in the same way that the expression $\langle \nabla_X Y, Z\rangle$ is tensorial in Z, so that $\nabla_X Y$ actually is defined pointwise by this formula.

2. Consider the metric

$$ds^2 = \frac{dx^2 + dy^2}{x^2 + y^2}$$

on the plane minus the origin, $\mathbb{R}^2 \backslash \{(0,0)\}$.

(a) Compute the covariant derivatives

$$\nabla_{\partial_x}\partial_x, \quad \nabla_{\partial_x}\partial_y, \quad \nabla_{\partial_y}\partial_x, \text{ and } \quad \nabla_{\partial_y}\partial_y.$$

Solution: By the Koszul formula above, we have

$$\langle \nabla_{\partial_i}\partial_j, \partial_k\rangle = \frac{1}{2}\big(\partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ik}\big),$$

so that the Christoffel symbols are

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}\big(\partial_i g_{jl} - \partial_l g_{ij} + \partial_j g_{il}\big).$$

Since the metric is diagonal and both components are the same function $f(x,y) = \frac{1}{x^2+y^2}$, we have

$$\Gamma_{ij}^k = \frac{1}{2f}\big(\delta_{jk}\partial_i f - \delta_{ij}\partial_k f + \delta_{ik}\partial_j f\big).$$

Thus

$$\begin{aligned}
\Gamma_{11}^1 &= \frac{x^2+y^2}{2}\partial_x(x^2+y^2)^{-1} = -\frac{x}{x^2+y^2} \\
\Gamma_{12}^1 &= \frac{x^2+y^2}{2}\partial_y(x^2+y^2)^{-1} = -\frac{y}{x^2+y^2} \\
\Gamma_{22}^1 &= -\frac{x^2+y^2}{2}\partial_x(x^2+y^2)^{-1} = \frac{x}{x^2+y^2} \\
\Gamma_{11}^2 &= -\frac{x^2+y^2}{2}\partial_y(x^2+y^2)^{-1} = \frac{y}{x^2+y^2} \\
\Gamma_{12}^2 &= \frac{x^2+y^2}{2}\partial_x(x^2+y^2)^{-1} = -\frac{x}{x^2+y^2} \\
\Gamma_{22}^2 &= \frac{x^2+y^2}{2}\partial_y(x^2+y^2)^{-1} = -\frac{y}{x^2+y^2}
\end{aligned}$$

From the formula $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$, we have

$$\nabla_{\partial_x}\partial_x = \frac{-x\partial_x + y\partial_y}{x^2+y^2}, \qquad \nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = \frac{-y\partial_x - x\partial_y}{x^2+y^2}, \qquad \nabla_{\partial_y}\partial_y = \frac{x\partial_x - y\partial_y}{x^2+y^2}.$$

(b) Compute the curvature

$$\langle R(\partial_x,\partial_y)\partial_x, \partial_y\rangle = \langle \nabla_{\partial_y}\nabla_{\partial_x}\partial_x - \nabla_{\partial_x}\nabla_{\partial_y}\partial_x + \nabla_{[\partial_x,\partial_y]}\partial_x, \partial_y\rangle.$$

Conclude that the space is flat.

Solution: We have

$$\begin{aligned}
\nabla_{\partial_y}\nabla_{\partial_x}\partial_x &= \nabla_{\partial_y}\left(\frac{-x\partial_x + y\partial_y}{x^2+y^2}\right) \\
&= \partial_y(x^2+y^2)^{-1}(-x\partial_x + y\partial_y) + \frac{1}{x^2+y^2}\partial_y + \frac{1}{x^2+y^2}(-x\nabla_{\partial_y}\partial_x + y\nabla_{\partial_y}\partial_y) \\
&= -\frac{2y}{(x^2+y^2)^2}(-x\partial_x + y\partial_y) + \frac{\partial_y}{x^2+y^2} + \frac{1}{x^2+y^2}\left(-x\frac{-y\partial_x - x\partial_y}{x^2+y^2} + y\frac{x\partial_x - y\partial_y}{x^2+y^2}\right) \\
&= \frac{1}{(x^2+y^2)^2}\big((2xy + xy + yx)\partial_x + (-2y^2 + x^2 + y^2 + x^2 - y^2)\partial_y\big) \\
&= \frac{4xy\partial_x + (2x^2 - 2y^2)\partial_y}{(x^2+y^2)^2}
\end{aligned}$$

and

$$\begin{aligned}
\nabla_{\partial_x}\nabla_{\partial_y}\partial_x &= \nabla_{\partial_x}\left(\frac{-y\partial_x - x\partial_y}{x^2+y^2}\right) \\
&= -\frac{2x}{(x^2+y^2)^2}(-y\partial_x - x\partial_y) - \frac{1}{x^2+y^2}\partial_y + \frac{1}{x^2+y^2}(-y\nabla_{\partial_x}\partial_x - x\nabla_{\partial_x}\partial_y) \\
&= \frac{2xy\partial_x + (2x^2 - x^2 - y^2)\partial_y}{(x^2+y^2)^2} + \frac{1}{(x^2+y^2)^2}\big(-y(-x\partial_x + y\partial_y) - x(-y\partial_x - x\partial_y)\big) \\
&= \frac{1}{(x^2+y^2)^2}\big((2xy + xy + xy)\partial_x + (x^2 - y^2 - y^2 + x^2)\partial_y\big) \\
&= \frac{4xy\partial_x + (2x^2 - 2y^2)\partial_y}{(x^2+y^2)^2}.
\end{aligned}$$

Therefore $R(\partial_x,\partial_y)\partial_x = 0$, so in particular $\langle R(\partial_x,\partial_y)\partial_x,\partial_y\rangle = 0$. If the curvature is zero, the metric must be flat.

(c) Can you find explicit coordinates (u,v) such that $ds^2 = du^2 + dv^2$? Hint: start by expressing the metric in polar coordinates.

Solution: In polar coordinates, we have $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$, so that the given metric becomes

$$ds^2 = \frac{dr^2 + r^2 d\theta^2}{r^2} = \frac{dr^2}{r^2} + d\theta^2.$$

To get this to look like Euclidean coordinates, we want a new coordinate u so that $du^2 = \frac{dr^2}{r^2}$, i.e., $du = \frac{dr}{r}$. Integrating this, we get $u = \ln r$. So the coordinate transformation is given by $x = e^u \cos v$, $y = e^u \sin v$.

3. Consider a paraboloid in $\mathbb{R}^3$ given by $z = x^2 + y^2 = r^2$.

   (a) What is the metric induced in polar coordinates (r, θ)?
   Solution: The flat metric in $\mathbb{R}^3$ is

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + dz^2.$$

   With $z = r^2$, we have $dz = 2r dr$, so that

$$ds^2 = \left(1 + 4r^2\right) dr^2 + r^2 d\theta^2.$$

   (b) Compute the geodesic equations. This will be much quicker if you use the Lagrangian $L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j$ and the Euler-Lagrange equations $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \frac{\partial L}{\partial x^i}$, though you can do it any way you want.
   Solution: Using the Lagrangian approach, we have

$$L = \frac{1}{2}(1 + 4r^2)\dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2.$$

   Thus the geodesic equations are

$$\frac{d}{dt}\big((1 + 4r^2)\dot{r}\big) = 4r\dot{r}^2 + r\dot{\theta}^2$$

   and

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0.$$

   (c) Solve the geodesic equations explicitly; i.e., obtain an integral giving t in terms of r, then get an integral giving θ in terms of t. You do not need to evaluate the integrals explicitly.
   Solution: The second of the geodesic equations integrates to $r^2\dot{\theta} = L$ for some constant L. We also have the energy-conservation law

$$(1 + 4r^2)\dot{r}^2 + r^2\dot{\theta}^2 = 1.$$

   Plugging the angular conservation law into this, we obtain

$$(1 + 4r^2)\dot{r}^2 + \frac{L^2}{r^2} = 1.$$

   Now solving for $\dot{r}$, we get

$$\dot{r} = \sqrt{\frac{1 - \frac{L^2}{r^2}}{1 + 4r^2}},$$

   so that

$$r\sqrt{\frac{1 + 4r^2}{r^2 - L^2}}\, dr = dt,$$

which gives the integral

$$t = \int_{r_o}^{r} p\sqrt{\frac{1+4p^2}{p^2-L^2}}\,dp.$$

If we performed this integral and found r in terms of t, we would then have for θ the integral

$$\theta = \theta_o + L\int_0^t \frac{d\tau}{r(\tau)^2}.$$

(d) Prove, just using the integrals, that if the geodesic does not pass through the origin, then $r \to \infty$ as $t \to \pm\infty$, and in addition that $\theta \to \pm\infty$ as $t \to \pm\infty$. Conclude that any geodesic that does not pass through the origin wraps around the paraboloid infinitely many times.

Solution: First let us consider the time integral

$$t = \int_{r_o}^{r} p\sqrt{\frac{1+4p^2}{p^2-L^2}}\,dp.$$

We ask what must happen to r as $t \to \infty$. The two conceivable ways for the integral to diverge are, first, that $r \to L$ (which makes the integrand singular) or that $r \to \infty$. If $r \to L$, then the integral looks like

$$\int_r^L \frac{L\sqrt{1+4L^2}}{\sqrt{p^2-L^2}}\,dp;$$

however this integral is *finite*. Therefore we must actually have $r \to \infty$.

Furthermore, we can estimate how fast $t \to \infty$ as $r \to \infty$. We always have $\frac{p}{\sqrt{p^2-L^2}} \geq 1$ (with equality holding if and only if $L = 0$). Therefore we have

$$t \geq \int_{r_o}^{r} \sqrt{1+4p^2}\,dp \geq \int_{r_o}^{r} 2p\,dp = r^2 - r_o^2.$$

Inverting this, we have

$$r^2 \leq r_o^2 + t \Longrightarrow \frac{1}{r(t)^2} \geq \frac{1}{r_o^2+t}.$$

Now using this in the θ integral, we have

$$\theta - \theta_o = \int_0^t \frac{L\,d\tau}{r(\tau)^2} \geq L\int_0^t \frac{1}{r_o^2+\tau}\,d\tau = L\ln\left(\frac{r_o^2+t}{r_o^2}\right).$$

As $t \to \infty$, therefore, we have $\theta \to \pm\infty$ (depending on the sign of L). A similar analysis works as $t \to -\infty$, since the geodesic equation is symmetric under time-reversal.

If $L = 0$, this doesn't work, corresponding to a geodesic passing through the origin. (If $L \neq 0$, then $r \geq L$ for all t, so the geodesic cannot pass through the origin.)