

Math 6240 Final Exam Solutions

1. The “Sol geometry” (one of the eight geometries in Thurston’s Geometrization) is \mathbb{R}^3 with group operation

$$(x, y, z) \cdot (u, v, w) = (x + e^z u, y + e^{-z} v, z + w)$$

and a left-invariant metric such that ∂_x , ∂_y , and ∂_z are orthonormal at the identity $(0, 0, 0)$.

- (a) Find the left-invariant vector fields E_1 , E_2 , and E_3 of Sol.

Solution: With $(p, q, r) = (x, y, z) \cdot (u, v, w) = L_{(x,y,z)}(u, v, w)$, we have

$$(DL_{(x,y,z)})_{(u,v,w)} = \begin{pmatrix} \frac{\partial p}{\partial u} & \frac{\partial p}{\partial v} & \frac{\partial p}{\partial w} \\ \frac{\partial q}{\partial u} & \frac{\partial q}{\partial v} & \frac{\partial q}{\partial w} \\ \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} & \frac{\partial r}{\partial w} \end{pmatrix} = \begin{pmatrix} e^z & 0 & 0 \\ 0 & e^{-z} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, the left-invariant vector fields are

$$E_1(x, y, z) = (DL_{(x,y,z)})_{(0,0,0)}(\partial_x|_{(0,0,0)}) = \begin{pmatrix} e^z & 0 & 0 \\ 0 & e^{-z} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e^z \partial_x|_{(x,y,z)}$$

$$E_2(x, y, z) = (DL_{(x,y,z)})_{(0,0,0)}(\partial_y|_{(0,0,0)}) = e^{-z} \partial_y|_{(x,y,z)}$$

$$E_3(x, y, z) = (DL_{(x,y,z)})_{(0,0,0)}(\partial_z|_{(0,0,0)}) = \partial_z|_{(x,y,z)}$$

- (b) Write the left-invariant metric in (x, y, z) coordinates explicitly.

Solution: The basis $\{E_1, E_2, E_3\}$ is orthonormal. Thus for example we have $1 = \langle E_1, E_1 \rangle = \langle e^z \partial_x, e^z \partial_x \rangle$, so that $\langle \partial_x, \partial_x \rangle = e^{-2z}$, which implies $g_{11}(x, y, z) = e^{-2z}$. Similarly we can derive the rest of the metric components, and we get

$$ds^2 = e^{-2z} dx^2 + e^{2z} dy^2 + dz^2.$$

- (c) Compute the nonzero Christoffel symbols of the left-invariant metric in (x, y, z) coordinates.

Solution: By the standard formula, we have

$$\Gamma_{ij}^k = \sum_{l=1}^3 \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

Since the nonzero metric components are diagonal, we can simplify this to

$$\Gamma_{ij}^k = \frac{1}{2g_{kk}} (\delta_{jk} \partial_i g_{kk} + \delta_{ik} \partial_j g_{kk} - \delta_{ij} \partial_k g_{ii}).$$

Furthermore, since the metric components only depend on z , we can write

$$\Gamma_{ij}^k = \frac{1}{2g_{kk}} ((\delta_{jk} \delta_{i3} + \delta_{ik} \delta_{j3}) \partial_z g_{kk} - \delta_{ij} \delta_{k3} \partial_z g_{ii}).$$

So we will have nonzero components only in the following cases: $j = k \neq 3$ and $i = 3$; $i = k \neq 3$ and $j = 3$ (which by symmetry is the same as the first); $i = j \neq 3$ and $k = 3$. These components are

$$\begin{aligned}\Gamma_{13}^1 &= \frac{1}{2g_{11}} \partial_z g_{11} = -1, \\ \Gamma_{23}^2 &= \frac{1}{2g_{22}} \partial_z g_{22} = 1, \\ \Gamma_{11}^3 &= -\frac{1}{2g_{33}} \partial_z g_{11} = e^{-2z}, \\ \Gamma_{22}^3 &= -\frac{1}{2g_{33}} \partial_z g_{22} = -e^{2z}.\end{aligned}$$

- (d) Compute the nonzero curvature components R_{ijkl} of the left-invariant metric in (x, y, z) coordinates.

Solution: We have

$$R_{ijkl} = \sum_{m=1}^3 g_{lm} \left(\partial_j \Gamma_{ik}^m - \partial_i \Gamma_{jk}^m + \sum_{p=1}^3 \Gamma_{ik}^p \Gamma_{jp}^m - \Gamma_{jk}^p \Gamma_{ip}^m \right).$$

Since this is a three-dimensional manifold, there are at most six independent curvature components: R_{1212} , R_{2323} , R_{3131} , R_{1231} , R_{2331} , and R_{1223} .

We have

$$\begin{aligned}
R_{1212} &= g_{22} \left(\partial_2 \Gamma_{11}^2 - \partial_1 \Gamma_{12}^2 + \sum_{p=1}^3 \Gamma_{11}^p \Gamma_{2p}^2 - \Gamma_{21}^p \Gamma_{1p}^2 \right) \\
&= e^{2z} \Gamma_{11}^3 \Gamma_{23}^2 \\
&= 1 \\
R_{2323} &= g_{33} \left(\partial_3 \Gamma_{22}^3 - \partial_2 \Gamma_{32}^3 + \sum_{p=1}^3 \Gamma_{22}^p \Gamma_{3p}^3 - \Gamma_{32}^p \Gamma_{2p}^3 \right) \\
&= \partial_z \Gamma_{22}^3 - \Gamma_{32}^2 \Gamma_{22}^3 \\
&= -2e^{2z} - (-e^{2z}) \\
&= -e^{2z} \\
R_{3131} &= g_{11} \left(\partial_1 \Gamma_{33}^1 - \partial_3 \Gamma_{13}^1 + \sum_{p=1}^3 \Gamma_{33}^p \Gamma_{1p}^1 - \Gamma_{13}^p \Gamma_{3p}^1 \right) \\
&= e^{-2z} (-\partial_z(-1) - (\Gamma_{13}^1)^2) \\
&= -e^{-2z} \\
R_{1231} &= g_{11} \left(\partial_2 \Gamma_{13}^1 - \partial_1 \Gamma_{23}^1 + \sum_{p=1}^3 \Gamma_{13}^p \Gamma_{2p}^1 - \Gamma_{23}^p \Gamma_{1p}^1 \right) \\
&= 0 \\
R_{2331} &= g_{11} \left(\partial_3 \Gamma_{23}^1 - \partial_2 \Gamma_{33}^1 + \sum_{p=1}^3 \Gamma_{23}^p \Gamma_{3p}^1 - \Gamma_{33}^p \Gamma_{2p}^1 \right) \\
&= 0 \\
R_{1223} &= g_{33} \left(\partial_2 \Gamma_{12}^3 - \partial_1 \Gamma_{22}^3 + \sum_{p=1}^3 \Gamma_{12}^p \Gamma_{2p}^3 - \Gamma_{22}^p \Gamma_{1p}^3 \right) \\
&= 0.
\end{aligned}$$

2. The *divergence* of a vector field X on a Riemannian manifold is a function $\operatorname{div} X$ defined to be the trace of the linear operator $Y \mapsto \nabla_Y X$.

(a) If $\{E_1, \dots, E_n\}$ is an orthonormal basis with $X = \sum_{k=1}^n X^k E_k$, show that

$$\operatorname{div} X = \sum_{k=1}^n E_k(X^k) - \sum_{j=1}^n \sum_{k=1}^n \langle [E_j, E_k], E_k \rangle X^j. \quad (1)$$

Solution: For an orthonormal basis, we can compute the trace as

$$\begin{aligned}
\operatorname{div} X &= \sum_{k=1}^n \langle \nabla_{E_k} X, E_k \rangle \\
&= \sum_{k=1}^n \sum_{j=1}^n \langle \nabla_{E_k} (X^j E_j), E_k \rangle \\
&= \sum_{k=1}^n \sum_{j=1}^n E_k(X^j) \langle E_j, E_k \rangle + X^j \langle \nabla_{E_k} E_j, E_k \rangle \\
&= \sum_{k=1}^n E_k(X^k) - \sum_{k=1}^n \sum_{j=1}^n \langle [E_j, E_k], E_k \rangle.
\end{aligned}$$

(b) If $\{x^1, \dots, x^n\}$ is a coordinate system with $X = \sum_{k=1}^n X^k \partial_k$, show that

$$\operatorname{div} X = \sum_{k=1}^n \partial_k X^k + \sum_{j=1}^n \sum_{k=1}^n \Gamma_{kj}^k X^j. \quad (2)$$

Solution: Observe first that

$$\nabla_{\partial_i} X = \nabla_{\partial_i} (X^k \partial_k) = \partial_i(X^k) \partial_k + X^k \nabla_{\partial_i} \partial_k = \partial_i(X^k) \partial_k + X^j \Gamma_{ij}^k \partial_k.$$

To obtain the trace, we want the operator in the form $A(\partial_i) = A_i^k \partial_k$; the trace is then $\sum_{k=1}^n A_k^k$. In this case, we have

$$\operatorname{div} X = \partial_k(X^k) + X^j \Gamma_{kj}^k.$$

(Here we used the summation convention.)

It is important to observe that the trace *cannot* be computed in coordinates using the formula

$$\sum_{k=1}^n \langle \nabla_{\partial_k} X, \partial_k \rangle.$$

First of all, such a formula for the trace is only valid in an *orthonormal* basis, where $A_k^k = \langle A(e_k), e_k \rangle$. In general this equality is not valid, and the correct one is $\sum_{k=1}^n A_k^k$. The other problem with the formula above is that it is not invariant; it depends on choice of coordinates (a clue for this is that it involves summing over two lower indices instead of one upper and one lower index).

If we wanted such a formula for the trace, we could use the inverse metric:

$$\operatorname{div} X = g^{ij} \langle \nabla_{\partial_i} X, \partial_j \rangle.$$

The above formula is the correct version.

- (c) For the metric $ds^2 = G(x, y)^2(dx^2 + dy^2)$, compute the divergence of a vector field $X = u \partial_x + v \partial_y$ in two ways: first do it in coordinates using (2), then do it in the orthonormal basis $E_1 = \frac{1}{G} \partial_x$, $E_2 = \frac{1}{G} \partial_y$ using (1). Check that both equations give the same result.

Solution: Using the coordinate formula, we have

$$\operatorname{div} X = \partial_x(u) + \partial_y(v) + u(\Gamma_{11}^1 + \Gamma_{21}^2) + v(\Gamma_{12}^1 + \Gamma_{22}^2).$$

To obtain the Christoffel symbols, we use

$$\begin{aligned}\Gamma_{11}^1 &= \frac{1}{2g_{11}}(\partial_1 g_{11}) = \frac{G_x}{G} \\ \Gamma_{12}^1 &= \frac{1}{2g_{11}}(\partial_1 g_{12} + \partial_2 g_{11} - \partial_1 g_{12}) = \frac{G_y}{G} \\ \Gamma_{12}^2 &= \frac{1}{2g_{22}}(\partial_1 g_{22} + \partial_2 g_{12} - \partial_2 g_{12}) = \frac{G_x}{G} \\ \Gamma_{22}^2 &= \frac{1}{2g_{22}}(\partial_2 g_{22}) = \frac{G_y}{G}.\end{aligned}$$

Thus the divergence is

$$\operatorname{div} X = u_x + v_y + \frac{2G_x u}{G} + \frac{2G_y v}{G}. \quad (3)$$

Using the orthonormal basis formula, we first write

$$X = GuE_1 + GvE_2.$$

Then formula (1) gives

$$\operatorname{div} X = E_1(Gu) + E_2(Gv) - Gu\langle [E_1, E_2], E_2 \rangle - Gv\langle [E_2, E_1], E_1 \rangle.$$

To proceed, we need the bracket:

$$[E_1, E_2] = \left[\frac{1}{G} \partial_x, \frac{1}{G} \partial_y \right] = -\frac{G_x}{G^3} \partial_y + \frac{G_y}{G^3} \partial_x = \frac{G_y}{G^2} E_1 - \frac{G_x}{G^2} E_2.$$

Then the formula becomes

$$\begin{aligned}\operatorname{div} X &= \frac{1}{G} \partial_x(Gu) + \frac{1}{G} \partial_y(Gv) + Gu \frac{G_x}{G^2} + Gv \frac{G_y}{G^2} \\ &= \frac{1}{G} (G_x u + Gu_x) + \frac{1}{G} (G_y v + Gv_y) + \frac{G_x u}{G} + \frac{G_y v}{G} \\ &= u_x + v_y + \frac{2G_x u}{G} + \frac{2G_y v}{G},\end{aligned}$$

which agrees with the coordinate computation (3).

3. For a general k -form ω expressed in coordinates as

$$\omega = \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

on a manifold, the $(k+1)$ -form $d\omega$ is defined as

$$d\omega = \frac{\partial \omega_{i_1 \dots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

It can be checked (but you don't have to) that this is actually a coordinate-invariant operation.

(a) Prove that in general, $d(d\omega) = 0$ for any k -form ω .

Solution: We have

$$d^2\omega = \frac{\partial^2 \omega_{i_1 \dots i_k}}{\partial x^m \partial x^j} dx^m \wedge dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

If we switch the dummy variables j and m , we don't change anything.

$$d^2\omega = \frac{\partial^2 \omega_{i_1 \dots i_k}}{\partial x^j \partial x^m} dx^j \wedge dx^m \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

Now we can use the fact that mixed partials commute

$$\frac{\partial^2 \omega_{i_1 \dots i_k}}{\partial x^j \partial x^m} = \frac{\partial^2 \omega_{i_1 \dots i_k}}{\partial x^m \partial x^j}$$

along with the antisymmetry of the wedge product

$$dx^j \wedge dx^m = -dx^m \wedge dx^j$$

to obtain

$$d^2\omega = -d^2\omega,$$

which implies $d^2\omega = 0$.

(b) For a general 1-form in \mathbb{R}^3 , given by $\omega = f(x, y, z)dx + g(x, y, z)dy + h(x, y, z)dz$, compute $d\omega$ from the formula above. What familiar vector calculus operation does it look like?

Solution:

$$\begin{aligned} d\omega &= f_x dx \wedge dx + f_y dy \wedge dx + f_z dz \wedge dx + g_x dx \wedge dy + g_y dy \wedge dy + g_z dz \wedge dy \\ &\quad + h_x dx \wedge dz + h_y dy \wedge dz + h_z dz \wedge dz \\ &= (h_y - g_z)dy \wedge dz + (f_z - h_x)dz \wedge dx + (g_x - f_y)dx \wedge dy. \end{aligned}$$

With the identifications $dy \wedge dz \mapsto \partial_x$, $dz \wedge dx \mapsto \partial_y$, and $dx \wedge dy \mapsto \partial_z$, this is the formula for the curl of the corresponding vector field $X = f\partial_x + g\partial_y + h\partial_z$.

- (c) Write down the most general 2-form α in \mathbb{R}^3 , and compute $d\alpha$. What familiar vector calculus operation does it look like?

Solution: There are only three linearly independent 2-forms in \mathbb{R}^3 (from the general formula $\binom{n}{k} = \binom{3}{2} = 3$). They are $dx \wedge dy$, $dy \wedge dz$, and $dz \wedge dx$. So the most general 2-form is

$$\alpha = p dy \wedge dz + q dz \wedge dx + r dx \wedge dy$$

for functions p , q , and r .

We now have

$$\begin{aligned} d\alpha &= p_x dx \wedge dy \wedge dz + p_y dy \wedge dy \wedge dz + p_z dz \wedge dy \wedge dz \\ &\quad + q_x dx \wedge dz \wedge dx + q_y dy \wedge dz \wedge dx + q_z dz \wedge dz \wedge dx \\ &\quad + r_x dx \wedge dx \wedge dy + r_y dy \wedge dx \wedge dy + r_z dz \wedge dx \wedge dy \\ &= p_x dx \wedge dy \wedge dz + q_y dy \wedge dz \wedge dx + r_z dz \wedge dx \wedge dy \\ &= (p_x + q_y + r_z) dx \wedge dy \wedge dz. \end{aligned}$$

Again, with the identification $\alpha \mapsto p\partial_x + q\partial_y + r\partial_z$, we see that $d\alpha$ is the divergence (multiplied by the volume form).

- (d) What familiar vector calculus identity does $d(d\omega) = 0$ generalize?

Solution: In this case, it says that $\operatorname{div} \operatorname{curl} X = 0$ for any vector field X .

4. A *Killing field* on a Riemannian manifold (not the Cambodian kind) is defined as a vector field X for which $\mathcal{L}_X g = 0$ (here g is the metric).

- (a) Using the definition of the Lie derivative of a tensor,

$$(\mathcal{L}_X g)(U, V) = X(g(U, V)) - g([X, U], V) - g(U, [X, V]),$$

show that X is a Killing field iff

$$\langle \nabla_U X, V \rangle + \langle \nabla_V X, U \rangle = 0 \text{ for all vectors } U \text{ and } V. \quad (4)$$

Solution: This is straightforward, using metric compatibility and no-torsion properties of the connection:

$$\begin{aligned} \mathcal{L}_X g(U, V) &= X\langle U, V \rangle - \langle [X, U], V \rangle - \langle U, [X, V] \rangle \\ &= \langle \nabla_X U, V \rangle + \langle U, \nabla_X V \rangle - \langle \nabla_X U - \nabla_U X, V \rangle - \langle U, \nabla_X V - \nabla_V X \rangle \\ &= \langle \nabla_U X, V \rangle + \langle U, \nabla_V X \rangle. \end{aligned}$$

Thus we have $\mathcal{L}_X g \equiv 0$ iff $\mathcal{L}_X g(U, V) = 0$ for all U and V , which happens iff (4) holds.

- (b) Suppose coordinates are chosen so that $X = \frac{\partial}{\partial x^1}$. Show that X is a Killing field iff the metric in these coordinates has

$$\frac{\partial g_{jk}}{\partial x^1} = 0 \text{ for all indices } j \text{ and } k.$$

Solution: Using (4), we know ∂_{x^1} is a Killing field iff $\langle \nabla_{\partial_j} \partial_1, \partial_k \rangle + \langle \nabla_{\partial_k} \partial_1, \partial_j \rangle = 0$ for all indices j and k . Thus we have, since coordinate vector fields commute:

$$\begin{aligned} \langle \nabla_{\partial_j} \partial_1, \partial_k \rangle + \langle \nabla_{\partial_k} \partial_1, \partial_j \rangle &= \langle \nabla_{\partial_1} \partial_j, \partial_k \rangle + \langle \partial_j, \nabla_{\partial_1} \partial_k \rangle \\ &= \partial_1 \langle \partial_j, \partial_k \rangle \\ &= \frac{\partial g_{jk}}{\partial x^1}. \end{aligned}$$

Hence $\frac{\partial}{\partial x^1}$ is a Killing field iff the components of the metric do not depend on x^1 .

- (c) If M is a Lie group with left-invariant metric and left-invariant vector fields E_i satisfying $[E_i, E_j] = \sum_{k=1}^n c_{ijk} E_k$, show that the fields E_i are Killing fields iff $c_{ijk} + c_{ikj} = 0$ for all i, j , and k .

Solution: Again using (4), we have

$$\begin{aligned} \langle \nabla_{E_j} E_i, E_k \rangle + \langle \nabla_{E_k} E_i, E_j \rangle &= \langle [E_j, E_i], E_k \rangle + \langle [E_k, E_i], E_j \rangle + E_i \langle E_j, E_k \rangle \\ &= -c_{ijk} - c_{ikj} + 0 \end{aligned}$$

Thus E_i is a Killing field iff $c_{ijk} + c_{ikj} = 0$ for all k and j . So we will have all E_i being Killing fields iff $c_{ijk} + c_{ikj} = 0$ for all i, j , and k , i.e., if and only if the metric is actually bi-invariant.