The Mother of All Integral Review Sheets (MAIRS)

Mathematics 2300

October 8, 2006

Formulas you'll want to memorize:

- Standard integrals (all should have "$+C$" of course)

$$\int x^k\,dx = \frac{x^{k+1}}{k+1} \quad (\text{if } k \neq -1) \qquad \int x^{-1}\,dx = \ln|x| \qquad \int e^x\,dx = e^x$$

$$\int \sin x\,dx = -\cos x \qquad \int \cos x\,dx = \sin x \qquad \int \sec^2 x\,dx = \tan x$$

$$\int \sec x \tan x\,dx = \sec x \qquad \int \sec x\,dx = \ln|\sec x + \tan x| \qquad \int \tan x\,dx = \ln|\sec x|$$

The following can be derived using trigonometric substitutions.

$$\int \frac{1}{x^2+1}\,dx = \arctan x \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x$$

- Trigonometric formulas

$$\cos^2 x + \sin^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x$$

$$\sin 2x = 2\sin x \cos x \qquad \cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

- Hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}, \text{ etc.}$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x + C \qquad \int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + C$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

How do you decide what technique to try?

- Use a u-substitution if...
  - you see a function of a function, and at least one is nonalgebraic,
    - Examples: $\int e^{\sqrt{x}}\,dx$ (use $u = \sqrt{x}$) and $\int e^x \sin(e^x)\,dx$ (use $u = e^x$) and $\int \sin x\sqrt{1+\cos x}\,dx$ (use $u = \cos x$) should all be approached by first u-substituting the inner function.
    - However, $\int (1+x^2)^{-3/2}\,dx$ is a composition of two algebraic functions, so substituting the inner function does not necessarily work in this case.
  - you have an algebraic function of an algebraic function, *and* you see the derivative of the inner function. Example: substitute $u = 1+x^2$ for $\int \frac{x}{\sqrt{1+x^2}}\,dx$ but not for $\int \frac{1}{\sqrt{1+x^2}}\,dx$.
- Use integration by parts if...
  - you see two unrelated functions being multiplied, like $\int x\cos x\,dx$ or $e^x \sin x\,dx$, but *not* $\int x^{-1/2}e^{\sqrt{x}}\,dx$ since the two functions are related (there's a composition, and the derivative of the inner function is being multiplied).
  - you see a logarithmic or inverse trigonometric function by itself, like $\int \sinh^{-1} x\,dx$ or $\int \ln x\,dx$ or $\int \arctan x\,dx$.
  - you want to derive a reduction formula for trigonometric functions, like $\int \sin^n x\,dx$ or $\int \sec^n x\,dx$.
- Use trigonometric identities and either a double-angle formula or a reduction formula if...
  - you have an even power of sines or cosines, like $\int \sin^2 x\,dx$ or $\int \cos^6 x\,dx$,
  - you have an odd power of secants or tangents, like $\int \sec^5 x\,dx$ or $\int \tan^3 x\,dx$.

  (If you have powers of trigonometric functions not in this form, then u-substitutions will work much more easily.)

- Use trigonometric substitution if...

  you see anything of the form

  - $(a^2 - x^2)^p$ (then let $x = a\sin\theta$)
  - $(a^2 + x^2)^p$ (then let $x = a\tan\theta$)
  - $(x^2 - a^2)^p$ (then let $x = a\sec\theta$)

  for any power p. Examples: $\dfrac{1}{(x^2-7)^{5/2}}\,dx$ or $\dfrac{1}{(x^2+5)^2}\,dx$.

- Use partial fractions if...

  you see a ratio of polynomials with distinct factors in the denominator.

Standard techniques:

- How to do u-substitution:

  - Choose u as the inside of some composed function.
  - Figure out du in terms of x and dx.
  - Write x in terms of u, and write dx in terms of u and du. Everything must be in terms of u before you continue!
  - If the integral you get is not simpler than the one you started with, you chose the wrong u-substitution. Start over.

- How to do integration by parts:

  - Make sure your integral is a product of two simple functions; there should be *no* compositions of simple functions. (If there are, you probably want to do substitution instead.)
  - Choose u and dv based on the mnemonic *LIATE*, which stands for "Logarithmic, Inverse (trigonometric or hyperbolic), Algebraic, Trigonometric or hyperbolic, Exponential." The u function should be whatever appears *first* in this list. You might remember this using "Luke Is All Too Easy."
  - Remember that 1 counts as algebraic! For example, you find $\int \ln x\,dx$ by setting $u = \ln x$ and $dv = dx$.
  - Once you have u and dv, find du and v. (Differentiate u and integrate dv.) Use the formula
    $$\int u\,dv = uv - \int v\,du.$$
  - If you end up with a harder integral than you started with, then *you messed up*. Either choose u and dv differently, or accept that integration by parts is not the right technique.

- If you end up with an integral that's just as difficult as what you started with, then you may have to do integration by parts *again*. Either everything will cancel out and you'll end up back where you started (bad), or things will not cancel and you'll be able to solve for the desired integral. For example, to find something $\int e^{ax} \sin bx \, dx$, you have to use integration by parts twice, then solve for the integral. The same trick applies for reduction formulas for trigonometric functions.
- If you end up with an integral that's easier than what you start with, keep doing integration by parts. Again and again.

- How to do trigonometric integrals (all of these techniques work also for hyperbolic integrals):
  - Many trigonometric integrals can be done using substitution and standard identities.
    * If you have a product of sines and cosines, and either one is raised to a non-negative odd power, then a substitution will work. For example, $\sin^2 x \cos^3 \; dx$ can be done by using the substitution $u = \sin x$ and the trig identity $\cos^2 x = 1 - \sin^2 x$.
    * If you have a product of tangents and secants with the secant raised to an even power, then substituting $u = \tan x$ will work. For example, $\sec^4 x \tan^3 x \, dx$ can be done using this substitution and the identity $\sec^2 x = 1 + \tan^2 x$.
    * If you have a product of tangents and secants with the secant raised to an odd power *and* the tangent raised to an odd power, then the substitution $u = \sec x$ will work. For example, $\int \sec^3 x \tan x$ can be done using this substitution and the identity $\sec^2 x = \tan^2 x + 1$.

    In general, if you forget the rules above, you can *always* try a substitution first. If it works, great. If it doesn't work, proceed to the next step...
  - If the above does not work, then you'll have to use either a double-angle formula or a reduction formula. First reduce everything to powers of the same functions; if you have sines and cosines combined, reduce everything to one or the other. If you have secants and tangents combined, reduce everything to one or the other. You can either...
    * use the double-angle formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

    * or use a reduction formula (assuming one is given to you), for example to reduce $\int \sin^n x \, dx$ to $\int \sin^{n-2} x \, dx$.

    Double-angle formulas tend to be quicker when dealing with small powers (2 or 4) while reduction formulas tend to be quicker when dealing with higher powers (6 or above). Theoretically, though, either method can be used.

- How to do trigonometric substitutions (all of this applies equally well to hyperbolic functions):
  - Use a trigonometric substitution when you have an integral containing a quadratic expression raised to a power: $(ax^2 + bx + c)^p$. Complete the square to get

$$ax^2 + bx + c = a\left(x + \frac{b}{a}\right)^2 + c - \frac{b^2}{4a}.$$

    You may want to use a substitution $u = x + b/a$ to simplify this.
  - You *must* use either the identity $\sin^2\theta + \cos^2\theta = 1$ or the identity $\sec^2\theta = \tan^2\theta + 1$ to simplify the integral. You're using this method on things of the form $(a^2 - x^2)^p$, or $(a^2 + x^2)^p$, or $(x^2 - a^2)^p$. If your choice of $x = a\sin\theta$ or $x = a\tan\theta$ or $x = \sec\theta$ does not lead to you using one of the above identities, then *you did it incorrectly.* For example, if you have $\int \frac{x^3}{\sqrt{5 + x^2}}\, dx$, you must substitute $x = \sqrt{5}\tan\theta$. Only this substitution will lead to

$$\sqrt{5 + x^2} = \sqrt{5 + 5\tan^2\theta} = \sqrt{5(1 + \tan^2\theta)} = \sqrt{5\sec^2\theta} = \sqrt{5}\sec\theta.$$

    If your constants don't match up (and you don't get to use one of the identities above), then *fix it.*
  - Remember to change *every* x to something involving θ. That means finding dx as well! If you're going to choose $x = 3\sec\theta$, then you must incorporate $dx = 3\sec\theta\tan\theta\, d\theta$ in your new integral, or it's *totally wrong.*
  - Once you've finished integrating, write your final answer in terms of the original variable. Don't leave your answer as $\sin(\arctan x)$; instead, use triangles and "SOH-CAH-TOA" to figure out: if $\tan\theta = x$, what is $\sin\theta$ in terms of x?

- How to do partial fractions:
  - If the degree of the numerator is greater than or equal to the degree of the denominator, use polynomial long division. Partial fractions *only* works when the degree of the numerator is less than the degree of the denominator.
  - Factor the denominator using algebra. You will obtain a product of terms: each one will give a different partial fraction.
    * $(x - c)$: write $\frac{A}{x - c}$.
    * $(x - c)^k$ for $k > 1$: write $\frac{A_1}{x - c} + \frac{A_2}{(x - c)^2} + \cdots + \frac{A_k}{(x - c)^k}$.
    * $(x^2 + ax + b)$ irreducible: write $\frac{Ax + B}{x^2 + ax + b}$.
    * $(x^2 + ax + b)^k$ for $k > 1$: write $\frac{A_1 x + B_1}{x^2 + ax + b} + \frac{A_2 x + B_2}{(x^2 + ax + b)^2} + \cdot \frac{A_k x + B_k}{(x^2 + ax + b)^k}$.

When you are done, the number of different functions you should have (each one multiplied by an unknown constant) must be the same as the degree of the denominator.

  - Get a common denominator and match the numerators. In simple cases, you can plug in special values of x which make all but one term zero, and use this to find the constants A, B, etc. In more complicated cases, you'll have to multiply out and solve the linear system of equations.
  - Plug the constants back in to the partial fraction expression, and integrate each term:
    * To integrate $(x-c)^k$, use the power rule.
    * To integrate something involving $(x^2+ax+b)^k$, complete the square and use either a u-substitution (if you have a numerator that works as du) or a trigonometric (tangent) substitution if not.

Checking your answer:

It's very easy to make little mistakes that change your answer completely. This is most common when doing integration by parts (forgetting to distribute) or partial fractions. If you have time, make sure you check your answers.

- You can always check an integral by computing the derivative of your answer. If you used u-substitution, your derivative should involve the chain rule. If you used integration by parts, your derivative should involve the product rule.
- If you've done a trigonometric substitution and used "SOH-CAH-TOA" to simplify your answer, it will be much easier to check your work by differentiating.
- You can always check your work with partial fractions by actually getting a common denominator, multiplying out, and verifying that you have the right numerator (before integrating).

Things to not memorize (but be able to derive them):

- Hyperbolic identities like $\cosh(x-y) = \cosh x\cosh y - \sinh x\sinh y$ or $\sinh(2x) = 2\sinh x\cosh x$.
- Derivatives of other hyperbolic functions like $\operatorname{sech} x$ and hyperbolic inverses like $\tanh^{-1} x$.
- Formulas for the inverse hyperbolic functions, such as $\sinh^{-1} x = \ln(x+\sqrt{x^2+1})$.
- Reduction formulas for trigonometric functions, such as $\int \sin^n x\,dx$ in terms of $\int \sin^{n-2} x\,dx$, as well as for hyperbolic functions (they work exactly the same way).
- Simplification formulas for compositions like $\sin(\arctan x)$ or $\operatorname{sech}(\sinh^{-1} x)$.